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A free boundary problem stemmed from combustion theory. Part II: Stability, instability and bifurcation results

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Abstract

We deal with a free boundary problem, depending on a real parameter λ, in a infinite strip in $\mathbb{R}^2$, providing stability, instability and bifurcation.

Keywords and Phrases: Free boundary problems; Linearization; Fully nonlinear elliptic and parabolic systems; Stability; Bifurcation

1. Introduction

In this paper we continue the study of the system, set in the infinite strip $\mathbb{R} \times (-l, l)$ ($l \in \mathbb{R}_+$), in the unknowns Θ, S and ξ:

$$\begin{aligned}
&\Theta_t(t,\eta,y) = \Delta\Theta(t,\eta,y), && t \geqslant 0,\ |y| < l,\ \eta < \xi(t,y),\\
&\Theta(t,\eta,y) = 1, && t \geqslant 0,\ |y| < l, \eta \geqslant \xi(t,y),\\
&S_t(t,\eta,y) = \Delta S(t,\eta,y) - \lambda\Delta\Theta(t,\eta,y), && t \geqslant 0,\ |y| < l,\ \eta \neq \xi(t,y),
\end{aligned}$$

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$$
\begin{aligned}
&[\Theta_\nu]\big(t,\eta(t,y),y\big) \\
&\quad + \exp\big(S\big(t,\eta(t,y),y\big)\big) = 0, && t \geqslant 0,\ |y| < l, \\
&\big[(S-\lambda\Theta)_\nu\big]\big(t,\eta(t,y),y\big) = 0, && t \geqslant 0,\ |y| < l, \\
&[\Theta]\big(t,\eta(t,y),y\big) = [S]\big(t,\eta(t,y),y\big) = 0, && t \geqslant 0,\ |y| < l, \\
&D_y\Theta(t,\eta,l) = D_yS(t,\eta,l) = 0, && t \geqslant 0,\ \eta \neq \xi(t,l), \\
&D_y\Theta(t,\eta,-l) = D_yS(t,\eta,-l) = 0, && t \geqslant 0,\ \eta \neq \xi(t,-l), \\
&D_y\eta(t,\pm l) = 0, && t \geqslant 0.
\end{aligned}
\tag{1.1}
$$

Here $\Delta = D_\eta^2 + D_y^2$ and with the symbol $[\,\cdot\,]$ we denote the jump at $\eta = \xi(t,y)$. At any point (η, y) with $\eta = \xi(t,y)$, we denote by $\nu = (\nu_1, \nu_2)$ the unit normal vector to the surface $\eta = \xi(t,y)$ with $\nu_1 > 0$.

We recall that problem (1.1) admits a planar travelling wave solution given by $(\Theta(t,x), S(t,x), \xi(t,y)) = (\Theta^0(x+t), S^0(x+t), -t)$, where

$$
\Theta^0(x) = \begin{cases} e^x, & x < 0, \\ 1, & x \geqslant 0, \end{cases} \qquad S^0(x) = \begin{cases} \lambda x e^x, & x < 0, \\ 0, & x \geqslant 0. \end{cases} \tag{1.2}
$$

In [8] we have proved existence–uniqueness results and regularity properties for the solution to problem (1.1) for initial data close to the travelling wave. Here we prove stability, instability results and bifurcation of nonplanar travelling waves.

One of the basic problems concerned with (1.1), pointed out in [1], consists in establishing whether the TW solution is stable or unstable for 2D-disturbances. A formal study made by Sivashinsky in [11], in the case of the whole space (i.e., $l = +\infty$), showed that there exists a (negative) critical value of λ (say λ_c) such that the TW should be orbitally stable for $\lambda \in (\lambda_c, 1)$ and orbitally unstable for $\lambda > 1$. In [5] and [3] a rigorous proof of instability for $\lambda > 1$ and stability for $\lambda = 0$ is given. Up to now, to the author's knowledge, the case where $\lambda \in (\lambda_c, 1]$ is still a challenging (from a mathematical viewpoint) open problem.

Here we give an answer to the question of the stability of the TW in the case of the strip. Denote by A_l the set

$$
A_l = (-\infty, \tilde{\lambda}) \cup \left(1 + \frac{\pi^2}{l^2}, +\infty\right), \tag{1.3}
$$

$\tilde{\lambda}$ being the supremum of the set

$$
\begin{aligned}
\Lambda := \Big\{&\lambda \in \big(-2-2\sqrt{3}, -16/3\big]: \\
&\big[\big(k_1(\lambda)-1\big)^{1/2} l/\pi, \big(k_2(\lambda)-1\big)^{1/2} l/\pi\big] \cap \mathbb{N}\setminus\{0\} \neq \emptyset\Big\},
\end{aligned}
$$

if $\Lambda \neq \emptyset$, and $\tilde{\lambda} = -2-2\sqrt{3}$ otherwise. Here

$$
\begin{aligned}
&k_j(\lambda) := k\big(\xi_j(\lambda)\big)/\lambda, \\
&\xi_j(\lambda) = \frac{8 - 3\lambda + (-1)^j\sqrt{9\lambda^2 + 48\lambda}}{16}, \quad j = 1, 2
\end{aligned}
\tag{1.4}
$$

and

$$k(\xi) = -16\xi^3 - 8(\lambda - 2)\xi^2 - (\lambda^2 - 8\lambda + 4)\xi + \lambda^2.$$

Our main results about stability can be collected in the following theorem.

Theorem A. *If $\lambda \in A_l$ the TW solution* (1.2) *to problem* (1.1)*, and the front are unstable with respect to smooth and sufficiently small* 2*D-perturbations, both in the weighted and in the nonweighted case. If $\lambda \in \mathbb{R} \setminus \overline{A_l}$, the TW solution* (1.2) *is orbitally stable in the weighted case. The solution to problem* (1.1) *converges to a translate of the TW solution* (*namely, there exists $r_\infty \in \mathbb{R}$ such that the solution to problem* (1.1) *converges to $(\Theta^0(x + r_\infty + t), S^0(x + r_\infty + t), t + r_\infty)$ as t tends to $+\infty$*).

To prove Theorem A we transform our problem into an equivalent fully nonlinear problem as we did in [8]. We get the system

$$\begin{cases} D_t\mathbf{u}(t,\cdot) = \mathcal{L}\mathbf{u}(t,\cdot) + \mathcal{F}(\mathbf{u}(t,\cdot)), & t > 0, \\ \mathcal{B}\mathbf{u}(t,\cdot) = \mathcal{G}(\mathbf{u}(t,\cdot)), & t > 0, \end{cases} \tag{1.5}$$

where $\mathcal{L}$ is the second-order differential operator

$$\mathcal{L}\mathbf{u} = (\Delta v - v_x, \Delta w - w_x - \lambda\Delta v, \Delta h + h_x), \quad \mathbf{u} = (v, w, h),$$

and the boundary differential operator $\mathcal{B} = (B_0, B_1, B_2)$ is given by

$$\begin{cases} (B_0\mathbf{u})(y) = \lambda v(0,y) - w(0,y) + h(0,y), & y \in [-l,l], \\ (B_1\mathbf{u})(y) = \lambda v(0,y) + \lambda v_x(0,y) - w_x(0,y) - h_x(0,y), & y \in [-l,l], \\ (B_2\mathbf{u})(y) = v(0,y) + h(0,y) - v_x(0,y), & y \in [-l,l]. \end{cases}$$

The nonlinear operator $\mathcal{F}$ is given by

$$\begin{aligned} \mathcal{F}\big(\mathbf{u}(t,\cdot)\big) &= \mathcal{F}_0\big(\mathbf{u}(t,\cdot)\big) \\ &\quad - \frac{\Delta v(t,0,\cdot) - v_x(t,0,\cdot) + f_1(\mathbf{u}(t,\cdot))(0,\cdot)}{1 - v(t,0,\cdot) + v_x(t,0,\cdot)}\Psi\big(\mathbf{u}(t,\cdot)\big), \end{aligned}$$

for any $t \in \mathbb{R}^+$, where $\mathcal{F}_0(\mathbf{u}) = (f_1(\mathbf{u}), f_2(\mathbf{u}), f_3(\mathbf{u}))$ is given by

$$\begin{aligned} f_1(\mathbf{u}) &= \big(v_y(0,\cdot)\big)^2\big(\Theta^0_{xx} - v(0,\cdot)\Theta^0_{xxx} + v_{xx}\big) \\ &\quad + 2v_y(0,\cdot)\big(-v_y(0,\cdot)\Theta^0_{xx} + v_{xy}\big) \\ &\quad + v_{yy}(0,\cdot)\big(-v(0,\cdot)\Theta^0_{xx} + v_x\big), \\ f_2(\mathbf{u}) &= \big(v_y(0,\cdot)\big)^2\big(S^0_{xx} - v(0,\cdot)S^0_{xxx} + w_{xx}\big) \\ &\quad + 2v_y(0,\cdot)\big(-v_y(0,\cdot)S^0_{xx} + w_{xy}\big) \\ &\quad + v_{yy}(0,\cdot)\big(-v(0,\cdot)S^0_{xx} + w_x\big) - \lambda f_1(\mathbf{u}), \\ f_3(\mathbf{u}) &= \big(v_y(0,\cdot)\big)^2 h_{xx} - 2v_y(0,\cdot)h_{xy} - v_{yy}(0,\cdot)h_x, \end{aligned}$$

for any $\mathbf{u}:\overline{\Omega}_l \to \mathbb{R}^3$ smooth enough (here and in what follows $\Omega_l := (-\infty, 0) \times (-l, l)$), while $\Psi(\mathbf{u})$ and $\mathcal{G}(\mathbf{u})$ are defined by

$$\Psi(\mathbf{u}) = \bigl(-v(0,\cdot)\Theta^0_{xx} + v_x, -v(0,\cdot)S^0_{xx} + w_x, -h_x\bigr)$$

and

$$\mathcal{G}(\mathbf{u}) = \bigl(0, 0, g(\mathbf{u})\bigr), \quad g(\mathbf{u}) = 1 + h(0,\cdot) - \bigl(1 + \bigl(v_y(0,\cdot)\bigr)^2\bigr)^{-1/2} e^{h(0,\cdot)}.$$

The transformations we perform to get (1.5) from (1.1) transform the TW (1.2) into the null solution to (1.5). Consequently, now the problem of the stability for the TW (1.2) reads as the problem of the stability of the null solution to problem (1.5).

We study the spectrum of the realization of $\mathcal{L}$ in nonweighted and suitably weighted spaces of continuous functions (see the forthcoming Definition 2.2), determining the set of all the $\lambda \in \mathbb{R}$ such that the spectrum of L does not contain elements with positive real part. However, since 0 is an eigenvalue for every λ, the principle of the linearized stability for fully nonlinear equations cannot be applied, and we have to use a Lyapunov–Schmidt procedure to prove Theorem A.

As far as bifurcation is concerned, we show that the set $\{\lambda = 1 + \pi^2 n^2/l^2\colon n \in \mathbb{N} \setminus \{0\}\}$ consists of bifurcation points of branches of nonplanar 2D-travelling wave solutions to (1.1) bifurcating from the trivial branch (Θ^0, S^0). By 2D-travelling wave solutions we mean any triplet of functions (Θ, S, ξ), explicitly depending on y, defined by $\Theta(t,\eta,y) = \Theta^1(\eta + ct, y)$, $S(t,\eta,y) = S^1(\eta + ct, y)$ and $\xi(t,y) = -ct + \xi^1(y)$, for some $c > 0$, that solves (1.1).

It is immediate to see that if (Θ, S, ξ) is a nonplanar solution to (1.1) then the triplet (Θ^1, S^1, ξ^1) solves

$$\begin{aligned}
&c\Theta^1_\eta(\eta,y) = \Delta\Theta^1(\eta,y), && y \in [-l,l],\ \eta < \xi^1(y),\\
&\Theta^1(\eta,y) = 1, && y \in [-l,l],\ \eta \geqslant \xi^1(y),\\
&cS^1_\eta(\eta,y) = \Delta S^1(\eta,y) - \lambda\Delta\Theta^1(\eta,y), && y \in [-l,l],\ \eta \neq \xi^1(y),\\
&\bigl[\Theta^1_\nu\bigr]\bigl(\xi^1(y),y\bigr) + \exp\bigl(S^1\bigl(\xi^1(y),y\bigr)\bigr) = 0, && y \in [-l,l],\\
&\bigl[(S^1 - \lambda\Theta^1)_\nu\bigr]\bigl(\xi^1(y),y\bigr) = 0, && y \in [-l,l],\\
&[\Theta^1]\bigl(\xi^1(y),y\bigr) = [S^1]\bigl(\xi^1(y),y\bigr) = 0, && y \in [-l,l],\\
&D_y\Theta^1\bigl(\eta,(-1)^j l\bigr) = D_y S^1\bigl(\eta,(-1)^j l\bigr) = 0, && \eta \neq \xi^1((-1)^j l),\ j = 0,1,\\
&\xi^1_y(\pm l) = 0.
\end{aligned} \tag{1.6}$$

We transform problem (1.6) into an equivalent one using the technique of [8]. First we fix the boundary at $x = 0$ and then, introducing the new unknowns v and w defined by

$$\begin{aligned}
&\text{(i)} \quad v(x,y) = \Theta^1\bigl(x + \xi^1(y), y\bigr) - \Theta^0(x) - \Theta^0_x(x)\xi^1(y),\\
&\text{(ii)} \quad w(x,y) = S^1\bigl(x + \xi^1(y), y\bigr) - S^0(x) - S^0_x(x)\xi^1(y),
\end{aligned} \tag{1.7}$$

we transform problem (1.6) into the fully nonlinear problem

$$\begin{cases} (c-1)(\mathbf{U}^0_x - v(0,\cdot)\mathbf{U}^0_{xx} + S_3\mathbf{u}_x) = \mathcal{L}\mathbf{u} + \mathcal{F}_0(\mathbf{u}), & \text{in } \overline{\Omega}_l, \\ \mathcal{B}\mathbf{u} = \mathcal{G}(\mathbf{u}), & \text{in } [-l,l], \\ D_y\mathbf{u}(\cdot,\pm l) = \mathbf{0}, & \text{in}(-\infty,0], \end{cases} \tag{1.8}$$

where $\mathbf{U}^0 = (\Theta^0, S^0, 0)$ and $S_3\mathbf{u} = (v, w, -h)$.

We show that if $\lambda \in 1 + (\pi^2/l^2)\mathbb{N}^2$, then $\omega = 0$ is a double eigenvalue of the realization L of $\mathcal{L}$ in the space $X_0^\sharp(\Omega_l)$ (cf. Definition 2.2). So, we have a problem of bifurcation from a double eigenvalue. But we reduce it to bifurcation from a simple eigenvalue, using the translation invariance of problem (1.1) to reduce the dimension of the kernel of $\mathcal{L}$, and getting the unknown velocity c in terms of (v, w) to reduce the codimension of the range of $\mathcal{L}$.

To this new problem we successfully apply the Crandall–Rabinowitz theorem obtaining the bifurcation results below.

Theorem B. *For any $n \in \mathbb{N}\setminus\{0\}$ there exist $\delta_n \in \mathbb{R}_+$, four infinitely many differentiable functions $\mathbf{U}_n : (-\delta_n, \delta_n) \to C^{2+\alpha}((-\infty,0]\times[-l,l]) \times C^{2+\alpha}((-\infty,0]\times[-l,l]) \times C^{2+\alpha}([0,+\infty)\times[-l,l])$,*

$$\mathbf{U}_n(\sigma)(x,y) = e^{-x/2}\big(v_n(\sigma)(x,y), w_n(\sigma)(x,y), h_n(\sigma)(x,y)\big),$$
$$\forall\sigma \in (-\delta_n,\delta_n),\ (x,y)\in\overline{\Omega}_l,$$

$\lambda_n, c_n : (-\delta_n,\delta_n) \to \mathbb{R}$, $\xi_n^1 : (-\delta_n,\delta_n) \to \{f \in C^{2+\alpha}([-l,l]):\ f'(\pm l) = 0\}$, with $\mathbf{U}_n(\sigma)$ explicitly depending on y, such that, for any $\sigma \in (-\delta_n,\delta_n)$, the pair $(\Theta_n(\sigma), S_n(\sigma))$ given by

$$\Theta_n(\sigma)(t,\eta,y) = v_n(\sigma)\big(t, \eta - \xi_n(\sigma)(t,y), y\big),$$
$$\eta \leqslant \xi_n(\sigma)(t,y) := -c_n(\sigma)t + \xi_n(\sigma)(y),$$
$$S_n(\sigma)(t,\eta,y) = \begin{cases} w_n(\sigma)(t,\eta-\xi_n(\sigma)(t,y),y), & \eta \leqslant \xi_n(\sigma)(t,y), \\ h_n(\sigma)(t,\eta-\xi_n(\sigma)(t,y),y), & \eta > \xi_n(\sigma)(t,y), \end{cases}$$

is a solution to problem (1.1) *with $\lambda = \lambda_n(\sigma)$. Moreover, $\mathbf{U}_n(0) = \mathbf{U}^0$, where $\mathbf{U}^0$ is the TW solution given by* (1.2) (*with $\lambda - \lambda_n(\sigma)$*), *$\mathbf{U}_n(\sigma) \not\equiv \mathbf{U}^0$ for any $\sigma \neq 0$, $\lambda_n(0) = 1 + \pi^2 n^2/l^2$, $c_n(0) = 1$, and $\xi_n^1(0) \equiv 0$. Further, for any $n \in \mathbb{N}\setminus\{0\}$, $c_n(\sigma)$ is locally greater than 1 in a neighborhood of $\sigma = 0$ and there exists $n_0 = n_0(l) \in \mathbb{N}$ such that $\lambda_n''(0) > 0$ for any $n \geqslant n_0$.*

It would be interesting to study the stability properties of the bifurcated travelling waves. However, this would need a considerable technical effort which would make the paper much longer and heavier. Therefore the topic will not be considered here.

2. The function spaces

In this section we introduce the Banach spaces we deal throughout this paper. For notation convenience we use a bold style to denote any vector valued function.

According to the notations of the Introduction, we denote by Ω_l ($l \in \mathbb{R}_+$) the set $\Omega_l = (-\infty, 0) \times (-l, l)$.

Definition 2.1. For any $k \geqslant 1$ and any $l \in \mathbb{R}_+$, we denote by $C^k_{\partial y}([-l,l])$ the set of all the functions $f \in C^k([-l,l])$ such that $f'(\pm l) = 0$, and we endow it with the norm of $C^k([-l,l])$.

Definition 2.2. For any $l > 0$, the function space $X_k(\Omega_l)$ is defined as follows:

$$X_k(\Omega_l) = \Big\{\mathbf{f} \in C^k(\overline{\Omega}_l)\colon \lim_{x\to-\infty} D^\alpha \mathbf{f}(x,y) = \mathbf{0},\ \forall y \in [-l,l],\ \forall |\alpha| \leqslant [k]\Big\},\quad k \geqslant 0.$$

We endow the space $X_k(\Omega_l)$ with the norm of $C^k(\overline{\Omega}_l)$.

Moreover, for any $T > 0$ and any $\alpha \in (0,1)$ we denote by $\mathcal{X}_{\alpha/2,\alpha}(0,T,\Omega_l)$ and $\mathcal{X}_{1+\alpha/2,2+\alpha}(0,T,\Omega_l)$ the Banach spaces

$$\begin{aligned}
\mathcal{X}_{\alpha/2,\alpha}(0,T,\Omega_l) = \Big\{\mathbf{u}\colon\ & \mathbf{u}(t,\cdot) \in X_\alpha(\Omega_l),\ \forall t \in [0,T],\\
& \sup_{0<t<T} \big\|\mathbf{u}(t,\cdot)\big\|_{X_\alpha(\Omega_l)} < +\infty,\\
& \mathbf{u}(\cdot,x,y) \in C^{\alpha/2}\big([0,T]\big),\ \forall (x,y) \in \overline{\Omega}_l,\\
& \sup_{(x,y)\in\Omega_l} \big\|\mathbf{u}(\cdot,x,y)\big\|_{C^{\alpha/2}([0,T])} < +\infty\Big\},\\
\mathcal{X}_{1+\alpha/2,2+\alpha}(0,T,\Omega_l) = \big\{\mathbf{u}\colon\ & D_t^{\alpha_1} D_x^{\alpha_2} D_y^{\alpha_3}\mathbf{u} \in \mathcal{X}_{\alpha/2,\alpha}(0,T,\Omega_l),\\
& \text{for } 2\alpha_1 + \alpha_2 + \alpha_3 \leqslant 2\big\}.
\end{aligned}$$

They are normed by

$$\|\mathbf{u}\|_{\mathcal{X}_{\alpha/2,\alpha}(0,T,\Omega_l)} = \sup_{0<t<T} \big\|\mathbf{u}(t,\cdot)\big\|_{X_\alpha(\Omega_l)} + \sup_{(x,y)\in\Omega_l} \big[\mathbf{u}(\cdot,x,y)\big]_{C^{\alpha/2}([0,T])},$$

$$\|\mathbf{u}\|_{\mathcal{X}_{1+\alpha/2,2+\alpha}(0,T,\Omega_l)} = \sum_{2\alpha_1+\alpha_2+\alpha_3\leqslant 2} \big\|D_t^{\alpha_1} D_x^{\alpha_2} D_y^{\alpha_3}\mathbf{u}\big\|_{\mathcal{X}_{\alpha/2,\alpha}(0,T,\Omega_l)}.$$

Next we define the weighted spaces $X_k^\sharp(\Omega_l)$ ($k \in \mathbb{R}_+$), $\mathcal{X}^\sharp_{\alpha/2,\alpha}(0,T,\Omega_l)$ and $\mathcal{X}^\sharp_{1+\alpha/2,2+\alpha}(0,T,\Omega_l)$. $X_k^\sharp(\Omega_l)$ consists of all the functions $\mathbf{f} := (f, g_1, g_2)$ such that

$$(x,y) \mapsto \mathbf{f}^\sharp(x,y) = \big(e^{-x/2} f(x,y), e^{-x/2} g_1(x,y), e^{x/2} g_2(x,y)\big)$$

belongs to $C^k(\overline{\Omega}_l)$. We norm $X_k^\sharp(\Omega_l)$ by taking $\|\mathbf{f}\|_{X_k^\sharp(\Omega_l)} = \|\mathbf{f}^\sharp\|_{C^k(\overline{\Omega}_l)}$ for any $\mathbf{f} \in X_k^\sharp(\Omega_l)$ and any $k \in \mathbb{R}_+$.

The weighted spaces $\mathcal{X}^{\sharp}_{\alpha/2,\alpha}(0,T,\Omega_l)$ and $\mathcal{X}^{\sharp}_{1+\alpha/2,2+\alpha}(0,T,\Omega_l)$ are defined as the corresponding nonweighted ones with $X_k(\Omega_l)$ ($l \in \mathbb{R}_+$) everywhere replaced by $X^{\sharp}_k(\Omega_l)$. We norm them accordingly to the norm of $\mathcal{X}_{\alpha/2,\alpha}(0,T,\Omega_l)$ and $\mathcal{X}_{1+\alpha/2,2+\alpha}(0,T,\Omega_l)$.

Finally by $X^{\sharp}_{2+\alpha,\partial y}(\Omega_l)$ we denote the set of all the functions $\mathbf{u} \in X^{\sharp}_{2+\alpha}(\Omega_l)$ such that $D_y\mathbf{u}(\cdot,\pm l) = 0$. We endow it with the norm of $X^{\sharp}_{2+\alpha}(\Omega_l)$.

Remark 2.3. It is easy to check that $X^{\sharp}_k(\Omega_l)$ can be characterized as the space of all the continuously differentiable up to the $[k]$-order functions such that $D^\alpha f \in X^{\sharp}_0(\Omega_l)$ for any $|\alpha| \leqslant [k]$ and $D^\alpha f \in X^{\sharp}_{k-[k]}(\Omega_l)$ for any $|\alpha| = [k]$. Moreover, the norm

$$|||\mathbf{f}|||_{X^{\sharp}_k(\Omega_l)} = \sum_{|\alpha|\leqslant[k]} \|D^\alpha \mathbf{f}\|_{X^{\sharp}_0(\Omega_l)} + \sum_{|\alpha|=[k]} [D^\alpha \mathbf{f}]_{X^{\sharp}_{k-[k]}(\Omega_l)},$$

is equivalent to the norm $\|\cdot\|_{X^{\sharp}_k(\Omega_l)}$.

Definition 2.4. For any $\alpha \in (0,1)$, $l, T > 0$, $C^{(j+\alpha)/2,j+\alpha}([0,T]\times[-l,l])$ ($j = 1,2$) denote the usual parabolic Hölder spaces

$$\begin{aligned}
&C^{(j+\alpha)/2,j+\alpha}\big([0,T]\times[-l,l]\big)\\
&\quad = \Big\{\psi\colon\ \psi(t,\cdot) \in C^{j+\alpha}\big([-l,l]\big),\ \sup_{t\in[0,T]} \big\|\psi(t,\cdot)\big\|_{C^{j+\alpha}([-l,l])} < +\infty,\\
&\qquad\qquad \psi(\cdot,y) \in C^{(j+\alpha)/2}\big([0,T]\big),\\
&\qquad\qquad \sup_{y\in[-l,l]} \big\|\psi(\cdot,y)\big\|_{C^{(j+\alpha)/2}([0,T])} < +\infty\Big\},
\end{aligned}$$

endowed with the norm

$$\begin{aligned}
\|\psi\|_{C^{(j+\alpha)/2,j+\alpha}([0,T]\times[-l,l])} &= \sup_{t\in[0,T]} \big\|\psi(t,\cdot)\big\|_{C^{j+\alpha}([-l,l])}\\
&\quad + \sup_{y\in[-l,l]} \big[\psi(\cdot,y)\big]_{C^{(j+\alpha)/2}([0,T])}, \qquad j = 1,2.
\end{aligned}$$

3. Stability and instability results

In this section we are devoted to prove stability and instability results for the TW solution (1.2) to problem (1.1) (i.e., for the null solution to (1.5)). For this purpose we need to determine the values of the real parameter λ such that the spectrum of L (that we denote by $\sigma(L)$ in the nonweighted case, and by $\sigma^{\sharp}(L)$ in the other case) does not contain elements with positive real part.

We recall that in [8] we have proved that

$$\sigma(L)=\bigl\{\omega\in\mathbb{C}\colon\ \mathrm{Re}\,\omega\leqslant-(\mathrm{Im}\,\omega)^2\bigr\}\cup\sigma_{\mathrm{point}}(L):=\sigma_{\mathrm{cont}}(L)\cup\sigma_{\mathrm{point}}(L),$$
$$\sigma^\sharp(L)=\{\omega\in\mathbb{R}_-\colon\ \omega\leqslant-1/4\}\cup\sigma_{\mathrm{point}}(L):=\sigma^\sharp_{\mathrm{cont}}(L)\cup\sigma_{\mathrm{point}}(L),\qquad(3.1)$$

where

$$\sigma_{\mathrm{point}}(L)=\bigcup_{n\in\mathbb{N}}\bigl\{\omega\in\mathbb{C}\colon\ \det M_\lambda(\omega,n)=0\bigr\},$$
$$\det M_\lambda(\omega,n)=\mu_{1,n}(\mu_{1,n}+\mu_{2,n})+\frac{\lambda\mu_{1,n}^2}{\mu_{1,n}+\mu_{2,n}}-\frac{\lambda\pi^2n^2}{4l^2}\frac{1}{\mu_{1,n}+\mu_{2,n}},\qquad(3.2)$$

and

$$\mu_{j,n}:=\mu_j(\omega,n)=\frac{1}{2}\left(1+4\omega+\frac{\pi^2n^2}{l^2}\right)^{1/2}+(-1)^j\frac{1}{2},\quad j=1,2,\ n\in\mathbb{N}.$$

Observe that $\sigma^\sharp_{\mathrm{cont}}(L)$ does not contain elements with nonnegative real part, while $\sigma_{\mathrm{cont}}(L)$ contains 0 as unique element with nonnegative real part. Hence to determine the set of all $\lambda\in\mathbb{R}$ such that $\sigma(L)$ (respectively, $\sigma^\sharp(L)$) does not contain elements with positive real part we need to study the dispersion relation defined by the infinitely many equations $\det M_\lambda(\omega,n)=0$, $n\in\mathbb{N}\cup\{0\}$.

3.1. A thorough analysis of the dispersion relation

Theorem 3.1. 0 *belongs to* $\sigma_{\mathrm{point}}(L)$ *for any* $\lambda\in\mathbb{R}$. *Moreover*:

(i) $\sigma_{\mathrm{point}}(L)\cap\{\omega\in\mathbb{C}\colon\ \mathrm{Re}\,\omega>0\}\neq\emptyset$ *if and only if* $\lambda\in A_l$;
(ii) $\sigma_{\mathrm{point}}(L)\cap\{\omega\in\mathbb{C}\colon\ \mathrm{Re}\,\omega\geqslant0\}=\{0\}$ *if and only if* $\lambda\notin\overline{A}_l$.

Proof. We begin by observing that the equation $\det M_\lambda(\omega,0)=0$ admits, for any $\lambda\in\mathbb{R}$, 0 as a solution. Moreover, 0 is its unique solution with nonnegative real part if and only if $\lambda\in(-2-2\sqrt{3},+\infty)$.

We now consider the equations $\det M_\lambda(\omega,n)=0$ with $n\geqslant1$ and determine the set of $\lambda\in\mathbb{R}$ such that the previous equations admit solutions with nonnegative real part. Of course, we can merely deal with the case $\lambda\in(-2-2\sqrt{3},+\infty)$.

We first determine the solutions z_j, $j=1,2,3$, to the equation

$$z^3+\frac{1}{2}(\lambda-2)z^2-\lambda z+\frac{\lambda}{2}-\frac{\lambda\pi^2n^2}{2l^2}=0,$$

and then we solve the equation

$$\left(1+4\omega+\frac{\pi^2n^2}{l^2}\right)^{1/2}=z_j,\quad j=1,2,3.\qquad(3.3)$$

Let us set $z_j = \xi + i\eta$. Since we are looking for solutions to (3.3) with nonnegative real part, the pair (ξ, η) turns out to be a solution to

$$\begin{cases} \text{(a) } \xi^3 - 3\xi\eta^2 + \frac{1}{2}(\lambda - 2)(\xi^2 - \eta^2) - \lambda\xi + \frac{\lambda}{2} - \lambda\frac{\pi^2 n^2}{2l^2} = 0, \\ \text{(b) } \eta^3 - 3\xi^2\eta - (\lambda - 2)\xi\eta + \lambda\eta = 0, \\ \text{(c) } \xi \geqslant \left(1 + \frac{\pi^2 n^2}{l^2}\right)^{1/2}, \\ \text{(d) } \xi^2 - \eta^2 - 1 - \frac{\pi^2 n^2}{l^2} \geqslant 0. \end{cases} \tag{3.4}$$

Let us consider solutions $z \in \mathbb{C}$ with $\eta = \operatorname{Im} z \neq 0$. Replacing the expression of η^2 in terms of ξ^2 given by (3.4b) into (3.4a) and (3.4d), leads us to the following equivalent system:

$$\begin{cases} \text{(e) } 8\xi^3 + 4(\lambda - 2)\xi^2 + \frac{1}{2}\left(\lambda^2 - 8\lambda + 4\right)\xi - \frac{1}{2}\lambda^2 + \frac{1}{2}\lambda + \lambda\frac{\pi^2 n^2}{2l^2} = 0, \\ \text{(f) } \eta^2 - 3\xi^2 - (\lambda - 2)\xi + \lambda = 0, \\ \text{(g) } \xi \geqslant \left(1 + \frac{\pi^2 n^2}{l^2}\right)^{1/2}, \\ \text{(h) } 2\xi^2 + (\lambda - 2)\xi + 1 - \lambda + \frac{\pi^2 n^2}{l^2} \leqslant 0. \end{cases} \tag{3.5}$$

Getting $(\pi n/l)^2$ from (3.5e) and replacing it in (3.5h) we get

$$\operatorname{sgn}(\lambda)\left(8\xi^2 + (3\lambda - 8)\xi + 2 - 3\lambda\right) \geqslant 0, \quad \lambda \neq 0. \tag{3.6}$$

Consider the case $\lambda > 0$ and observe that $8\xi^2 + (3\lambda - 8)\xi + 2 - 3\lambda > 0$ for any $\xi \geqslant 1$. Consequently, any solution to (3.5e) satisfies also (3.6).

Let us study (3.5e) and denote by $h_{n,\lambda}$ its left-hand side. It is easy to see that $h_{n,\lambda}$ attains its minimum value at

$$\xi_\lambda = \frac{1}{6}(2 - \lambda) + \frac{1}{12}(\lambda^2 + 8\lambda + 4)^{1/2},$$

and $\xi_\lambda < 1$ for any $\lambda > 0$. Consequently, $h_{n,\lambda}(\xi) \geqslant h_{n,\lambda}(1) > 2$ for any $\xi \geqslant 1$ so that, if $\lambda > 0$, problem (3.5) does not admit any solution with $\eta \neq 0$.

In the case $\lambda = 0$ we see immediately that there is not any solution to (3.5) with $\eta \neq 0$.

Let us consider the case $\lambda < 0$ and the function $k_\lambda(\xi) = 8\xi^2 + (3\lambda - 8)\xi + 2 - 3\lambda$, $\xi \in \mathbb{R}$. Since $k_\lambda(\xi) > 0$ for any $\xi \geqslant 1$ when $\lambda \in (-16/3, 0)$, we deduce that (3.6) does not hold for these values of λ. Consequently, problem (3.5) does not admit any solution z with $\operatorname{Im} z \neq 0$ if $\lambda \in (-16/3, 0)$.

Suppose now that $\lambda \in (-2 - 2\sqrt{3}, -16/3]$. Then $3\xi^2 + (\lambda - 2)\xi - \lambda \geqslant 0$ for any $\xi \geqslant 1$. Consequently, any solution to (3.5e) and (3.5g) satisfies also (3.5f). Observe that, if $\lambda \in (-2 - 2\sqrt{3}, -16/3]$, then the function $h_{n,\lambda}$ is increasing in $[1, +\infty)$ for any $n \in \mathbb{N}$. Define the function m_1 by setting

$$m_1(\xi, \lambda) = 8\xi^3 + \frac{1}{2}(9\lambda - 16)\xi^2 + \frac{1}{2}(\lambda^2 - 8\lambda + 4)\xi - \frac{1}{2}\lambda^2,$$

for any $(\xi,\lambda)\in[1,+\infty)\times[-2-2\sqrt{3},-16/3]$. As is easily seen

$$h_{n,\lambda}\left(\left(1+\frac{\pi^2n^2}{l^2}\right)^{1/2}\right)=m_1\left(\left(1+\frac{\pi^2n^2}{l^2}\right)^{1/2},\lambda\right).$$

Observe that for any $\xi\geqslant 1$, the function $m_1(\xi,\cdot)$ is increasing and for any $\lambda\in(-2-2\sqrt{3},-16/3]$, the equation $m_1(\cdot,\lambda)=0$ admits a unique root $\xi(\lambda)\geqslant 2$. Equation (3.5a) admits a solution $\xi_*(\lambda)\geqslant 1+\pi^2n^2/l^2$ if and only if $1+\pi^2n^2/l^2\leqslant\xi(-2-2\sqrt{3})$. $\xi_*(\lambda)$ satisfies (3.6) if and only if it belongs to $[\xi_1(\lambda),\xi_2(\lambda)]$ or, equivalently, if and only if $1+\pi^2n^2/l^2\in[k_1(\lambda),k_2(\lambda)]$ (see (1.4)). A straightforward computation shows that the functions k_1 and k_2 are, respectively, increasing and decreasing for $\lambda\in[-2-2\sqrt{3},-16/3]$. Moreover, $k_1(-16/3)=k_2(-16/3)=7/6$ while $k_1(-2-2\sqrt{3})=1$, $k_2(-2-2\sqrt{3})=1+\sqrt{3}/4$. Hence, if $\pi^2n^2/l^2>\sqrt{3}/4$, then problem (3.5) admits no solution (ξ,η) with $\eta\neq 0$. If there exists $\lambda_0\in(-2-2\sqrt{3},-16/3]$ such that $1+\pi^2n^2/l^2\in[k_1(\lambda_0),k_2(\lambda_0)]$, then problem (3.5) admits solutions (ξ,η) for any $\lambda\in(-2-2\sqrt{3},\lambda_0]$.

Let us now study the real solutions to problem (3.3). We have to deal with the problem

$$\xi^3+\frac{1}{2}(\lambda-2)\xi^2-\lambda\xi+\frac{\lambda}{2}-\frac{\lambda\pi^2n^2}{2l^2}=0,\quad \xi\geqslant\left(1+\frac{\pi^2n^2}{l^2}\right)^{1/2},\tag{3.7}$$

for $n>0$. Observe that for any $\lambda\in(-2-2\sqrt{3},+\infty)$ the function R_n defined by the left-hand side of the differential equation in (3.7) is increasing for $\xi\geqslant 1$. In particular,

$$R_n\big[(1+\pi^2n^2l^{-2})^{1/2}\big]=m_2\big[(1+\pi^2n^2l^{-2})^{1/2}\big],$$

where $m_2(\xi)=\xi^3-\xi^2-\lambda\xi+\lambda=(\xi-1)(\xi+\sqrt{\lambda})(\xi-\sqrt{\lambda})$. Hence problem (3.7) admits no solution for any $n\in\mathbb{N}$, $n\geqslant 1$ if and only if $\lambda\in(-\infty,1+\pi^2/l^2)$. So we get the assertion. □

3.2. Instability

Let us now prove that if $\lambda\in A_l$ the null solution to problem (1.5) is unstable with respect to both nonweighted and weighted perturbations. For this purpose, we provide a backward solution to problem

$$\begin{cases} D_t\mathbf{w}(t,\cdot)=\mathcal{L}\mathbf{w}(t,\cdot)+\mathcal{F}(\mathbf{w}(t,\cdot)), & t\leqslant 0,\\ \mathcal{B}\mathbf{w}(t,\cdot)=\mathcal{G}(\mathbf{w}(t,\cdot)), & t\leqslant 0,\\ D_y\mathbf{w}(t,\cdot,\pm l)=\mathbf{0}, & t\leqslant 0,\end{cases}\tag{3.8}$$

which goes to 0 as $t\to-\infty$.

We will prove our results in the nonweighted case, since the proofs can be easily adapted to the weighted spaces.

First of all let us prove that if $\lambda \in A_l$, then there exists a suitable Jordan curve separating $\sigma_+(L) := \{\omega \in \mathbb{C}\colon \operatorname{Re}\omega > 0\}$ from the rest of the spectrum of L.

Lemma 3.2. *For any $b \in \mathbb{R}$, $\sigma_+(L) \cap \{\omega \in \mathbb{C}\colon \operatorname{Re}\omega > b\}$ consists of finitely many elements.*

Proof. Suppose that ω solves the equation $\det M_\lambda(\omega, n) = 0$ and $\operatorname{Re}\omega > b$. By squaring both sides of the equation we deduce that ω is a root of the polynomial Q_n defined by $Q_n(z) := z^3 + c_1(n)z^2 + c_2(n)z + c_3(n)$, where

$$c_1(n) \sim \frac{3}{4}\frac{\pi^2 n^2}{l^2}, \qquad c_2(n) \sim \frac{3}{16}\frac{\pi^4 n^4}{l^4}, \qquad c_3(n) \sim \frac{1}{64}\frac{\pi^6 n^6}{l^6},$$
$$\text{as } n \to +\infty.$$

Of course, a complex number ω with $\operatorname{Re}\omega > b$ solves the equation $Q_n(z) = 0$ if and only if $\omega - b$ is a solution, with positive real part, to $\overline{Q}_n(z) = 0$, where $\overline{Q}_n(z) = z^3 + c_1'(n)z^2 + c_2'(n)z + c_3'(n)$ and

$$c_1'(n) \sim \frac{3}{4}\frac{\pi^2 n^2}{l^2}, \qquad c_2'(n) \sim \frac{3}{16}\frac{\pi^4 n^4}{l^4}, \qquad c_3'(n) \sim \frac{1}{64}\frac{\pi^6 n^6}{l^6},$$
$$\text{as } n \to +\infty.$$

From the Routh–Hurwitz theorem (cf. [6, Chapter 2, Section 4]) we deduce that for sufficiently large n, $\overline{Q}_n$ does not admit roots with positive real part. This implies that $\sigma_{\text{point}}(L) \cap \{\omega \in \mathbb{C}\colon \operatorname{Re}\omega > b\}$ consists of finitely many elements. □

From Lemma 3.2 we deduce that if $\lambda \in A_l$, $\sigma_+(L)$ is a nonempty set consisting of finitely many points. Hence we can define the spectral projection P_+ associated to $\sigma_+(L)$ by

$$P^+ = \frac{1}{2\pi i}\int_\gamma R(\omega, L)\, d\omega, \tag{3.9}$$

where γ is a suitable simple and closed Jordan curve separating $\sigma_+(L)$ from the rest of the spectrum of L.

We are now in a position to prove the following result.

Theorem 3.3. *Suppose that $\lambda \in A_l$, and fix $0 < \omega_0 < \min\{\operatorname{Re}\omega\colon \omega \in \sigma_+(L)\}$. Then problem* (3.8) *has a nontrivial backward solution $\mathbf{u}$ such that $e^{-\omega_0 t}\mathbf{u} \in \mathcal{X}_{1+\alpha/2,2+\alpha}(-\infty, 0, \Omega_l)$ (see Definition* 2.2).

Proof. Fix $\mathbf{u}_0 \in P^+(X_0(\Omega_l))$ and define the set $\mathcal{Y}$ by

$$\mathcal{Y} = \Big\{\mathbf{u} \in \mathcal{X}_{1+\alpha/2,2+\alpha}(-\infty, 0, \Omega_l)\colon\ D_y\mathbf{u}(\cdot\,, \pm l) = 0,\ \mathbf{u}(0, \cdot) = \mathbf{u}_0,$$
$$\|e^{\omega_0 t}\mathbf{u}\|_{\mathcal{X}_{1+\alpha/2,2+\alpha}(-\infty,0,\Omega_l)} \leqslant \rho\Big\}.$$

Define the *nonlinear* operator Γ in $\mathcal{Y}$ by

$$\begin{aligned}\Gamma(\mathbf{u})(t,\cdot) = e^{tL}\mathbf{u}_0 &+ \int_0^t e^{(t-s)L}P^+\big[\mathcal{F}\big(\mathbf{u}(s,\cdot)\big) + \mathcal{L}\mathcal{N}g\big(\mathbf{u}(s,\cdot)\big)\big]\,ds \\ &+ \int_{-\infty}^t e^{(t-s)L}(I-P^+)\big[\mathcal{F}\big(\mathbf{u}(s,\cdot)\big) + \mathcal{L}\mathcal{N}g\big(\mathbf{u}(s,\cdot)\big)\big]\,ds \\ &- L\int_{-\infty}^t e^{(t-s)L}(I-P^+)\mathcal{N}g\big(\mathbf{u}(s,\cdot)\big)\,ds \\ &- L\int_0^t e^{(t-s)L}P^+\mathcal{N}g\big(\mathbf{u}(s,\cdot)\big)\,ds,\end{aligned}$$

for any $t \leqslant 0$ and any $\mathbf{u} \in \mathcal{Y}$, where $\mathcal{N}$ is the lifting operator defined at the beginning of Appendix A. Observe that, thanks to Lemma A.1, $\Gamma(\mathbf{u})$ is the (unique) solution in $\mathcal{X}_{1+\alpha/2,2+\alpha}(-\infty,0,\Omega_l)$ to problem (3.8) with $(\mathcal{F}(\mathbf{w}),\mathcal{G}(\mathbf{w}))$ being replaced by $(\mathcal{F}(\mathbf{u}),\mathcal{G}(\mathbf{u}))$, which satisfies the initial condition $\mathbf{v}(0,\cdot)=\mathbf{u}_0$, provided we show that $e^{-\omega_0 t}\mathcal{F}(\mathbf{u})$ and $e^{-\omega_0 t}g(\mathbf{u})$ belong to $\mathcal{X}_{\alpha/2,\alpha}(-\infty,0,\Omega_l)$ and $C^{1+\alpha/2,1+\alpha}((-\infty,0]\times[-l,l])$, respectively. Since, as is easily seen, any solution to (3.8) is a fixed point of the operator Γ, we will get the assertion of the theorem if we prove that Γ is a contraction mapping $\mathcal{Y}$ into itself. Straightforward computations, and the fact that $\mathcal{F}' \in C(B(0,\rho); L(X_{2+\alpha}(\Omega_l), X_\alpha(\Omega_l))) \cap \mathrm{Lip}(B(0,\rho); L(X_2(\Omega_l), X_0(\Omega_l)))$ and $g' \in C(B(0,\rho); L(X_{2+\alpha}(\Omega_l), C^{1+\alpha}([-l,l]))) \cap \mathrm{Lip}(B(0,\rho); L(X_2(\Omega_l), C([-l,l])))$ for suitable small $\rho > 0$, show that $\mathcal{F}(\mathbf{u}) \in \mathcal{X}_{\alpha/2,\alpha}(-\infty,0,\Omega_l)$ and $g(\mathbf{u}) \in C^{(1+\alpha)/2,1+\alpha}((-\infty,0]\times[-l,l])$ for any $\mathbf{u}\in\mathcal{Y}$. Moreover, there exists a positive function K_1 which goes to 0 as ρ tends to 0 such that

$$\begin{aligned}&\big\|e^{-\omega_0 t}\big[\mathcal{F}(\mathbf{u})-\mathcal{F}(\mathbf{v})\big]\big\|_{\mathcal{X}_{\alpha/2,\alpha}(-\infty,0,\Omega_l)} \\ &\quad + \big\|e^{-\omega_0 t}\big[g(\mathbf{u})-g(\mathbf{v})\big]\big\|_{C^{(1+\alpha)/2,1+\alpha}((-\infty,0]\times[-l,l])} \\ &\leqslant K_1(\rho)\big\|e^{-\omega_0 t}(\mathbf{u}-\mathbf{v})\big\|_{\mathcal{X}_{1+\alpha/2,2+\alpha}(-\infty,0,\Omega_l)},\end{aligned} \tag{3.10}$$

for any $\mathbf{u},\mathbf{v}\in\mathcal{Y}$.

From (3.10) and Lemma A.1, we can now show that Γ is a contraction in $\mathcal{Y}$ provided ρ and $\rho_0 \leqslant \rho$ are sufficiently small. Consequently, problem (3.8) admits a unique solution in $\mathcal{Y}$. □

Theorem 3.3 holds also in the weighted case and its proof can be obtained with slight changes from the proof of Theorem 3.3.

Thanks to Theorem 3.3, we can prove instability of the null solution to problem (1.5). In fact, using a well-known technique we can show that there exists a sequence of initial data $\mathbf{u}_{0,n} \in X_{2+\alpha}(\Omega_l)$ (respectively, $\mathbf{v}_{0,n} \in X^{\sharp}_{2+\alpha}(\Omega_l)$) ($n \in \mathbb{N}$) tending to 0 in $X_{2+\alpha}(\Omega_l)$ (respectively, in $X^{\sharp}_{2+\alpha}(\Omega_l)$) such that the solution $\mathbf{u}_n$ (respectively, $\mathbf{v}_n$) to problem (1.5) satisfying $\mathbf{u}_n(0,\cdot) = \mathbf{u}_{0,n}$ (respectively, $\mathbf{v}_n(0,\cdot) = \mathbf{v}_{0,n}$) does not converge to 0 in $X_0(\Omega_l)$ (respectively, in $X^{\sharp}_0(\Omega_l)$) as $n \to +\infty$.

Therefore we have

Theorem 3.4. *Suppose that $\lambda \in A_l$. Then the null solution to problem* (1.5) *is unstable both with respect to the $X_0(\Omega_l)$- and $X^{\sharp}_0(\Omega_l)$-norm.*

Coming back to our original problem (1.1), Theorem 3.4 implies that the TW solution (1.2) and the front $\xi(t) = -t$ are unstable with respect to 2D-perturbations for any $\lambda \in A_l$.

3.3. Stability in $X^{\sharp}_0(\Omega_l)$

In this section we prove that if $\lambda \notin \overline{A}_l$ (see (1.3)) then the null solution to problem (1.5) is stable in the $X^{\sharp}_{2+\alpha}(\Omega_l)$-norm. Unfortunately, we cannot apply immediately the Linearized Stability Principle in $X^{\sharp}_0(\Omega_l)$ to prove the stability results since, as pointed out in Theorem 3.1, $\omega = 0$ is an eigenvalue of L for any $\lambda \in \mathbb{R}$. However, we shall show that $\omega = 0$ is a semisimple eigenvalue of L. In particular, if $\lambda \notin \overline{A}_l$ then $\omega = 0$ is a simple eigenvalue of L and the kernel of L is spanned by the function $\mathbf{U}^0_x(x) = (e^x, \lambda(x+1)e^x, 0)$. Then we split the solution $\mathbf{u}$ to (1.5) as $\mathbf{u} = P_0\mathbf{u} + (I - P_0)\mathbf{u} := r(t)\mathbf{U}^0_x + \mathbf{w}$, where P_0 is the spectral projection associated with the eigenvalue $\omega = 0$, obtaining a system for the pair $(r, \mathbf{w})$, the second equation involving *only* the function $\mathbf{w}$. We successfully apply the Linearized Stability Principle to the equation for w, and then we solve the system.

We observe that for any $\mathbf{f} = (f, g_1, g_2) \in X^{\sharp}_0(\Omega_l)$, $R(\omega, L)\mathbf{f} = \mathbf{v}_1 + \mathbf{v}_2$, where $\mathbf{v}_1$ and $\mathbf{v}_2$ are given by

$$\begin{aligned}
\mathbf{v}_j(x,y) = {} & \frac{1}{2l}\sum_{n=0}^{+\infty} a_1(n)\cos\left(\frac{\pi n}{2l}y - \frac{\pi n}{2}\right)\int_{-\infty}^{0} e^{\mu_{j,n}t}\hat{\mathbf{f}}_j(t+x,n)\,dt \\
& + \frac{1}{2l}\sum_{n=0}^{+\infty} a_1(n)\cos\left(\frac{\pi n}{2l}y - \frac{\pi n}{2}\right)\int_{x}^{0} e^{\mu_{3-j,n}t}\hat{\mathbf{f}}_j(x-t,n)\,dt \\
& + \frac{1}{2l}e^{(2-j)x}\sum_{n=0}^{+\infty}\frac{\mathbf{a}_{2,j}(n)}{\det M_\lambda(\omega,n)}\cos\left(\frac{\pi n}{2l}y - \frac{\pi n}{2}\right)
\end{aligned}$$

$$
\begin{aligned}
&\times \int_{-\infty}^{0} e^{\mu_{1,n}(t+x)}\hat{g}_1(t,n)\,dt \\
&+ \frac{1}{2l}\sum_{n=0}^{+\infty}\frac{\mathbf{a}_{3,j}(n)}{\det M_\lambda(\omega,n)}\cos\left(\frac{\pi n}{2l}y - \frac{\pi n}{2}\right) \\
&\times \int_{-\infty}^{0} e^{\mu_{3-j,n}(t+x)}e^{(j-1)t}\hat{g}_2(t,n)\,dt \\
&+ \frac{1}{2l}e^{(2-j)x}\sum_{n=0}^{+\infty}\frac{\mathbf{a}_{4,j}(n)}{\det M_\lambda(\omega,n)}\cos\left(\frac{\pi n}{2l}y - \frac{\pi n}{2}\right) \\
&\times \int_{-\infty}^{0} te^{\mu_{1,n}(t+x)}\hat{f}(t,n)\,dt \\
&+ \frac{1}{2l}e^{(2-j)x}\sum_{n=0}^{+\infty}\frac{\mathbf{a}_{5,j}(n)}{\det M_\lambda(\omega,n)}\cos\left(\frac{\pi n}{2l}y - \frac{\pi n}{2}\right) \\
&\times \int_{-\infty}^{0} e^{\mu_{1,n}(t+x)}\hat{f}(t,n)\,dt,
\end{aligned}
\tag{3.11}
$$

$j = 1, 2$, $\mathbf{f}_1 = (f, g_1 - \lambda\Delta v, 0)$, $\mathbf{f}_2 = (0, 0, g_2)$, while $a_1(n) = \mu_{1,n} + \mu_{2,n}$, $a_2(n) = a_1(n)(\pi^2 n^2/(4l^2) - \mu_{1,n}^2)$ and

$$
\begin{aligned}
\mathbf{a}_{2,j}(n) &= \big(2-j, (2-j)\lambda a_1(n)\mu_{2,n}, (1-j)\mu_{1,n}\big), \\
\mathbf{a}_{3,j}(n) &= \Big(2-j, (2-j)\big(\lambda a_1(n)a_2(n) + \lambda\big(a_1(n)+1\big) + \mu_{1,n}\big), \\
&\qquad (j-1)\frac{\lambda}{2}a_1(n)\big(1-(1+2\omega)a_1(n)\big)\Big), \\
\mathbf{a}_{4,j}(n) &= \lambda\big((j-2)a_1(n)a_2(n), (j-2)\lambda a_1(n)a_2(n) + (\mu_{2,n}+1), \\
&\qquad (j-1)a_1(n)a_2(n)\mu_{2,n}\big), \\
\mathbf{a}_{5,j}(n) &= \big((2-j)\lambda a_1(n)a_2(n) + \mu_{2,n}, (2-j)\lambda^2 a_1(n)a_2(n) \\
&\qquad + (1-j)\lambda(1+\mu_{2,n}), (1-j)a_1^{-1}(n)\mu_{1,n}\big),
\end{aligned}
$$

for $j = 1, 2$.

In the following theorem we show that 0 is a semisimple eigenvalue of $R(\cdot, L)$ and we provide an explicit representation formula for the projection P_0.

Theorem 3.5. *For any $\lambda \in \mathbb{R}$, $\omega = 0$ is a semisimple eigenvalue of the operator L. In particular, if $\lambda \neq 1 + \pi^2 n^2/l^2$ for any $n \in \mathbb{N}$, then $\omega = 0$ is a simple eigenvalue* *of L and the corresponding eigenspace is spanned by the function $\mathbf{U}_x^0$. Moreover, $P_0\mathbf{u} = (T_0\mathbf{u})\mathbf{U}_x^0$, where*

$$\begin{aligned} T_0(v, w, h) = \Bigg[\frac{1}{2l} \int_{-\infty}^{0} dx \int_{-l}^{l} v(x, y)\, dy + \int_{-\infty}^{0} dx \int_{-l}^{l} w(x, y)\, dy \\ + \int_{-\infty}^{0} \left(\frac{1}{2l} e^x + 1 - \frac{1}{2l} \right) dx \int_{-l}^{l} h(x, y)\, dy \Bigg]. \end{aligned} \tag{3.12}$$

If $\lambda \in 1 + (\pi^2/l^2)\mathbb{N}^2$, then $\omega = 0$ is a double eigenvalue of L and the corresponding eigenspace is spanned by $\mathbf{U}_x^0$ and $\mathbf{U}^\lambda$, where

$$\begin{aligned} \mathbf{U}^\lambda(x, y) = \begin{pmatrix} e^{(\sqrt{\lambda}+1)x/2} \\ \left[\lambda + \frac{\sqrt{\lambda}}{2} - \frac{1}{2} + \left(\frac{\sqrt{\lambda}}{2} + \frac{\lambda}{2}\right)x\right] e^{(\sqrt{\lambda}+1)x/2} \\ \left(\frac{\sqrt{\lambda}}{2} - \frac{1}{2}\right) e^{(1-\sqrt{\lambda})x/2} \end{pmatrix} \\ \times \cos\left(\frac{\pi n}{2l} y - \frac{\pi n}{2} \right). \end{aligned} \tag{3.13}$$

The spectral projection onto $\operatorname{Ker} L$ *is given by*

$$P_0(\mathbf{u}) = T_0(\mathbf{u})\mathbf{U}_x^0 + T_\lambda(\mathbf{u})\mathbf{U}^\lambda, \tag{3.14}$$

where

$$\begin{aligned} T_\lambda(v, w, h) = \rho \Bigg\{ \sqrt{\lambda} \int_{-\infty}^{0} \left[1 - \left(\frac{\sqrt{\lambda}}{2} - \frac{1}{2} \right) x \right] e^{(1-\sqrt{\lambda})x/2}\, dx \\ \times \int_{-l}^{l} v(x, y) \cos\left(\frac{\pi n}{2l} y - \frac{\pi n}{2} \right) dy \\ + \int_{-\infty}^{0} e^{(1-\sqrt{\lambda})x/2}\, dx \int_{-l}^{l} w(x, y) \cos\left(\frac{\pi n}{2l} y - \frac{\pi n}{2} \right) dy \\ + \int_{-\infty}^{0} e^{(\sqrt{\lambda}+1)x/2}\, dx \int_{-l}^{l} h(x, y) \cos\left(\frac{\pi n}{2l} y - \frac{\pi n}{2} \right) dy \Bigg\}, \end{aligned} \tag{3.15}$$

and $\rho = \sqrt{\lambda}(\lambda + 2\sqrt{\lambda} - 2)^{-1} l^{-1}$.

Proof. To show that 0 is a semisimple eigenvalue, we have to check that 0 is a simple pole of the function $\omega \mapsto R(\omega, L)$. From Lemma 3.2 it follows that 0 is isolated in $\sigma(L)$.

To show that 0 is a simple pole of the resolvent, we observe that for any $\lambda \in \mathbb{R}$, 0 is a simple zero of the equation $\det M_\lambda(\cdot\,,0) = 0$ (see (3.2)) and $\det M_\lambda(0,n) \neq 0$ for $n \in \mathbb{N}$ unless $\lambda = 1 + \pi^2 n^2/l^2$ (in such a case 0 is still a simple zero). Moreover, as pointed out in [8, Theorem 4.3], the coefficients $a_1(n)$ and $(\det M_\lambda(\omega,n))^{-1}\mathbf{a}_{k,j}(n)$, $j = 1,2$, $k = 2,\ldots,5$, in (3.11) do not exceed $C_1 n^{-1}$, C_1 being a positive constant, independent of ω and n in a neighborhood of 0, for n sufficiently large. A straightforward computation shows that $|\det M_\lambda(\omega,n)| \geqslant C_2 n^2$, for any $\omega \in B(0,r)\setminus\{0\}$, any $n \geqslant n_0(r)$ and some positive constant C_2. This implies that the function $\omega \mapsto \omega R(\omega, L)$ has a removable singularity at $\omega = 0$.

Observe now that the dimension of $\operatorname{Ker} L$ does not exceed two. Indeed, the dispersion relation $\det M_\lambda(\omega,n) = 0$ admits $\omega = 0$ as solution if and only if $n = 0$ or $n = n_0$ and $\lambda = 1 + \pi^2 n_0^2/l^2$. This immediately implies that if $\lambda \neq 1 + \pi^2 n^2/l^2$ for any $n \in \mathbb{N} \cup \{0\}$ then 0 is a simple eigenvalue of L. In such a case, a straightforward computation shows that the kernel of L is spanned by $\mathbf{U}_x^0$ and that $P_0(\mathbf{u}) = (T_0\mathbf{u})\mathbf{U}_x^0$ for any $\mathbf{u} \in X_0^\sharp(\Omega_l)$.

Let $\lambda = 1 + \pi^2 n_0^2/l^2$ for some $n_0 \in \mathbb{N}$. Let $\mathbf{u} \in \operatorname{Ker} L$, and denote by $\mathbf{v} : \overline{\Omega}_{2l} \to \mathbb{C}$ the function defined by $\mathbf{v}(x,y) = \mathbf{u}(x, l + (-1)^{j-1}y)$ for any $x \leqslant 0$ and any $(-1)^j y \in [0, 2l]$, $j = 1,2$. As is easily seen $L\mathbf{v} = 0$ in Ω_{2l}. We recall that in [8, Theorem 4.3], we have proved that the Fourier coefficients of the function $y \mapsto \mathbf{v}(\cdot\,,y)$ (say $(\hat{v}(\cdot\,,n), \hat{w}(\cdot\,,n), \hat{h}(\cdot\,,n))$) solve the algebraic equations

$$M_\lambda(\omega,n)\big(\hat{v}(\cdot\,,n), \hat{w}(\cdot\,,n), \hat{h}(\cdot\,,n)\big) = (0,0,0), \quad n \in \mathbb{N}, \tag{3.16}$$

where $M_\lambda(\omega,n)$ is a 3×3 matrix whose determinant is given by the right-hand side of (3.2). Of course, if $n \notin \{0, n_0\}$, then (3.16) implies $(\hat{v}(\cdot\,,n), \hat{w}(\cdot\,,n), \hat{h}(\cdot\,,n)) = (0,0,0)$. Moreover, $M_\lambda(\omega,0)$ and $M_\lambda(\omega,n_0)$ have rank equal to two (see [8, Theorem 4.3]). Therefore, $(\hat{v}(\cdot\,,0), \hat{w}(\cdot\,,0), \hat{h}(\cdot\,,0))$ (respectively, $(\hat{v}(\cdot\,,n_0), \hat{w}(\cdot\,,n_0), \hat{h}(\cdot\,,n_0)))$) is determined up to a multiplicative constant C_0 (respectively, C_{n_0}). This implies that the kernel of L has geometric multiplicity equal to two.

Straightforward computations show that $\mathbf{U}_x^0$ and $\mathbf{U}^\lambda$ are eigenfunctions of L with eigenvalue $\omega = 0$ and that the operator defined by the right-hand side in (3.14) is a projection onto $\operatorname{Ker} L$ commuting with L. This implies that it coincides with P_0 (cf. [10, Lemma A.2.8]). □

We now project problem (1.5) into $P_0(X_0^\sharp(\Omega_l))$ and into $(I - P_0)(X_0^\sharp(\Omega_l))$, respectively. The system will be easily decoupled since

$$\mathcal{F}(\mathbf{u}) \equiv \mathcal{F}\big((I - P_0)\mathbf{u}\big), \qquad \mathcal{G}(\mathbf{u}) \equiv \mathcal{G}\big((I - P_0)\mathbf{u}\big), \qquad \mathcal{F}(P_0\mathbf{u}) \equiv 0,$$
$$\mathcal{G}(P_0\mathbf{u}) \equiv 0, \quad \mathbf{u} \in X_2^\sharp(\Omega_l).$$

We split the solution of the initial value problem $\mathbf{u}(0,\cdot)=\mathbf{u}_0$ for (1.5) as $\mathbf{u} = P_0\mathbf{u} + (I-P_0)\mathbf{u} = r(t)\mathbf{U}_x^0 + \mathbf{v}$, and observe that the pair $(r,\mathbf{v})$ solves the systems

$$\begin{cases} r'(t) = T_0[\mathcal{L}\mathbf{v}(t,\cdot)+\mathcal{F}(\mathbf{v}(t,\cdot))], & t\in[0,T],\\ r(0)=T_0[\mathbf{u}_0], \end{cases} \tag{3.17}$$

$$\begin{cases} D_t\mathbf{v} = \widetilde{\mathcal{L}}\mathbf{v} + (I-P_0)\mathcal{F}(\mathbf{v}), & \text{in } [0,T]\times\overline{\Omega}_l,\\ \mathcal{B}(\mathbf{v}) = \mathcal{G}(\mathbf{v}), & \text{in } [0,T]\times[-l,l],\\ D_y\mathbf{v}(\cdot,\pm l) = \mathbf{0}, & \text{in } [0,T]\times(-\infty,0],\\ \mathbf{v}(0,\cdot) = (I-P_0)\mathbf{u}_0, & \text{in } \overline{\Omega}_l, \end{cases} \tag{3.18}$$

where $\widetilde{\mathcal{L}} = (I-P_0)\mathcal{L}$.

We can now prove the following stability result.

Theorem 3.6. *Suppose that $\lambda\notin\overline{A}_l$. Then the null solution to problem* (1.5) *is stable in the $X^\sharp_{2+\alpha}(\Omega_l)$-norm. To be more precise for any $\omega_0\in(0,-\max\{\mathrm{Re}\,\omega\colon \omega\in\sigma_+(L)\})$ there exist $r>0$ and $R>0$ such that, for any $\mathbf{u}_0\in B(0,r)\subset X^\sharp_{2+\alpha}(\Omega_l)$ satisfying the compatibility conditions*

$$\mathcal{B}(\mathbf{u}_0) = \mathcal{G}(\mathbf{u}_0), \qquad B_0\big[\mathcal{L}\mathbf{u}_0 + \mathcal{F}(\mathbf{u}_0)\big] = 0, \qquad D_y\mathbf{u}_0(\cdot,\pm l) = \mathbf{0},$$

the solution $\mathbf{u} = (I-P_0)\mathbf{u} + r(t)\mathbf{U}_x^0$ to problem (1.5) *with initial datum $\mathbf{u}_0$ exists for $t\in[0,+\infty)$ and satisfies the following estimate*:

$$\big\|e^{\omega_0 t}(I-P_0)\mathbf{u}\big\|_{\mathcal{X}^\sharp_{1+\alpha/2,2+\alpha}(0,+\infty,\Omega_l)} \leqslant R\|\mathbf{u}_0\|_{X^\sharp_{2+\alpha}(\Omega_l)}. \tag{3.19}$$

Moreover, there exists $r_\infty\in\mathbb{R}$ such that $r_\infty = \lim_{t\to+\infty} r(t)$.

Proof. The proof follows the same ideas of the proof of Theorem 3.3 and, consequently, it is only sketched. Let us introduce the set $\mathcal{Y}$ defined by

$$\mathcal{Y} = \Big\{\mathbf{v}\in B\big([0,+\infty);X\big)\colon\ D_y\mathbf{v}(\cdot,\pm l)=0,\ \mathbf{v}(0,\cdot)=\mathbf{u}_0,\\ \|e^{\omega_0 t}\mathbf{v}\|_{\mathcal{X}^\sharp_{1+\alpha/2,2+\alpha}(0,+\infty,\Omega_l)} \leqslant \rho\Big\},$$

where B stays for bounded, and consider the *nonlinear* operator Γ defined by

$$\Gamma(\mathbf{v})(t,\cdot) = e^{tL}\mathbf{u}_0 + \int_0^t e^{(t-s)L}\big(\mathcal{F}\big(\mathbf{v}(s,\cdot)\big) + \widetilde{\mathcal{L}}\mathcal{N}g\big(\mathbf{v}(s,\cdot)\big)\big)\,ds \\ - L\int_0^t e^{(t-s)L}(I-P_0)\mathcal{N}g\big(\mathbf{v}(s,\cdot)\big)\,ds, \quad t\geqslant 0,$$

for any $\mathbf{v}\in\mathcal{Y}$. Arguing as in the proof of Theorem 3.3 and taking Lemma A.3 into account, it can be proved that the operator Γ is well defined and it is a contraction mapping in $\mathcal{Y}$ provided ρ and r are sufficiently small. This implies that the equation $\Gamma(\mathbf{v}) = \mathbf{v}$ admits a unique solution in $\mathcal{Y}$ that turns out to be a solution to problem (3.18) with $T = +\infty$.

An easy computation shows that $e^{\omega_0 t} T_0(\mathcal{L}\mathbf{v} + \mathcal{F}(\mathbf{v})) \in C^{\alpha/2}([0,+\infty))$. Consequently, problem (3.17) admits a unique solution $r \in C^{1+\alpha/2}([0,+\infty))$ given by

$$r(t) = T_0(\mathbf{u}_0) + \int_0^t T_0\big[\mathcal{L}\mathbf{v}(s,\cdot) + \mathcal{F}\big(\mathbf{v}(s,\cdot)\big)\big]\,ds.$$

Now, the function $\mathbf{u}(t,\cdot) = r(t)\mathbf{U}^0 + \mathbf{v}(t,\cdot)$, $t \geqslant 0$, is easily seen to solve problem (1.5) and to enjoy property (3.19). Moreover, $r(t)$ converges to the real number $T_0(\mathbf{u}_0) + \int_0^t T_0[\mathcal{L}\mathbf{v}(s,\cdot) + \mathcal{F}(\mathbf{v}(s,\cdot))]\,ds$ as t tends to $+\infty$. $\square$

Remark 3.7. Coming back to our original problem (1.1) the results in Theorem 3.6 read saying that the solution to the initial value problem associated with (1.1) exists in the time interval $[0,+\infty)$ for initial data sufficiently close to the TW wave in (1.2). Moreover, the functions $(t,x,y) \mapsto (\Theta(t, x-\xi(t,y), y), S(t, x-\xi(t,y), y))$ and $(t,y) \mapsto \xi(t,y) + t$ converge, respectively, to the TW function in (1.2) in the $X_\sharp^{2+\alpha}$-norm, and to $-r_\infty$ as $t \to +\infty$. Hence the solution to (1.1) converges to a translate of the TW (1.2); namely, it converges to the TW function $(\Theta^0(x + r_\infty + t), S^0(x + r_\infty + t), t + r_\infty)$ as t tends to $+\infty$.

4. Bifurcation results

In this section we prove that there exists a sequence of values of the real parameter λ giving raise to bifurcated branches of nonplanar travelling waves.

4.1. Some additional results on the linear operator L

We need some preliminaries to prove the bifurcation results.

Theorem 4.1. *Assume that $\lambda \in 1 + (\pi^2/l^2)\mathbb{N}^2$. Then the range of the operator*

$$\mathcal{K} : X^\sharp_{2+\alpha,\partial y}(\Omega_l) \to X^\sharp_\alpha(\Omega_l) \times C^{2+\alpha}_{\partial y}\big([-l,l]\big) \times \big(C^{1+\alpha}_{\partial y}\big([-l,l]\big)\big)^2,$$

defined by $\mathcal{K}\mathbf{u} = (\mathcal{L}\mathbf{u}, \mathcal{B}\mathbf{u})$, where $C^{1+\alpha}_{\partial y}([-l,l])$, $C^{2+\alpha}_{\partial y}([-l,l])$ and $X^\sharp_{2+\alpha,\partial y}(\Omega_l)$ are defined in Definitions 2.1 *and* 2.2*, has codimension* 2 *and consists of all the quadruplets* $(\mathbf{f}, k_0, k_1, k_2)$ *such that*

$$\mathcal{R}_1(\mathbf{f}, k_0, k_1, k_2) := \left(T_0\mathbf{f} + \int_{-l}^{l} \left(k_1(y) - k_0(y) + \frac{1}{2l}k_2(y)\right) dy\right) \times \big(\mathbf{U}^0_x, 0, 0, 0\big) = \mathbf{0}, \tag{4.1}$$

$$\mathcal{R}_2(\mathbf{f},k_0,k_1,k_2) := \Bigg[T_\lambda\mathbf{f} + \rho\int_{-l}^{l}\big(k_1(y)+\sqrt{\lambda}k_2(y)\big)\cos\left(\frac{\pi n}{2l}y - \frac{\pi n}{2}\right)dy - \left(\frac{\sqrt{\lambda}}{2}+\frac{1}{2}\right)\rho\int_{-l}^{l}k_0(y)\cos\left(\frac{\pi n}{2l}y-\frac{\pi n}{2}\right)dy\Bigg] \times (\mathbf{U}^\lambda,0,0,0) = \mathbf{0}. \tag{4.2}$$

Moreover, the restriction $\widetilde{\mathcal{K}}$ *of* $\mathcal{K}$ *to* $(I-P_0)(X^\sharp_{2+\alpha,\partial y}(\Omega_l))$ *is an isomorphism between* $(I-P_0)(X^\sharp_{2+\alpha,\partial y}(\Omega_l))$ *and* $(I-\mathcal{R}_1-\mathcal{R}_2)(X^\sharp_\alpha(\Omega_l)\times C^{2+\alpha}_{\partial y}([-l,l])\times C^{1+\alpha}_{\partial y}([-l,l])\times C^{1+\alpha}_{\partial y}([-l,l]))$, *where* P_0 *is defined by* (3.14) *and* (3.15).

Proof. Fix $\mathbf{f}\in X^\sharp_\alpha(\Omega_l)$, $k_0\in C^{2+\alpha}_{\partial y}([-l,l])$, $k_1,k_2\in C^{1+\alpha}_{\partial y}([-l,l])$, and consider the functional equation $(\mathcal{L}\mathbf{u},\mathcal{B}\mathbf{u}) = (\mathbf{f},k_0,k_1,k_2)$. Define the function $\mathbf{z}$ by setting $\mathbf{z} = (I-P_0)(\mathcal{M}(k_0,k_1)+\mathcal{N}k_2)$, where $\mathcal{N}$ is the lifting operator defined at the beginning of Appendix A. The operator $\mathcal{M}$ is defined by $\mathcal{M}(g_0,g_1) = M_0g_0+M_1g_1$, where $M_0g_0(x,y) = (\varphi_1(x)g_0(y),\varphi_2(x)g_0(y),0)$ and $M_1g_1(y) = (0,-Ng_1,0)$ for any $g_0,g_1\in C([-l,l])$, and N is the operator defined in (A.2) while $\varphi_1,\varphi_2\in C^\infty_0(\mathbb{R})$ are smooth functions which vanish outside $(-1,1)$ and satisfy the following conditions: $\varphi_1(0)=\varphi_1'(0)=1$, $\varphi_2(0)=\lambda-1$, $\varphi_2'(0)=2\lambda$. It is immediate to check that $\mathcal{M}\in L(C^{2+\alpha}_{\partial y}([-l,l])\times C^{1+\alpha}_{\partial y}([-l,l]), X^\sharp_{2+\alpha,\partial y}(\Omega_l))$, $\mathcal{B}\mathcal{M}(g_1,g_2) = (g_1,g_2,0)$ for any $g_1\in C^{2+\alpha}_{\partial y}([-l,l])$ and any $g_2\in C^{1+\alpha}_{\partial y}([-l,l])$.

Obviously, $\mathbf{z}\in(I-P_0)(X^\sharp_{2+\alpha,\partial y}(\Omega_l))$ and $\mathcal{B}\mathbf{z}=\mathbf{k}$, $\mathbf{k}=(k_0,k_1,k_2)$, since $\mathcal{B}P_0\equiv 0$. As is easily seen the equation $(\mathcal{L}\mathbf{u},\mathcal{B}\mathbf{u}) = (\mathbf{f},k_0,k_1,k_2)$ is solvable if and only if the equation $(\mathcal{L}\mathbf{w},\mathcal{B}\mathbf{w}) = (\mathbf{f}-\mathcal{L}\mathbf{z},\mathbf{0})$ admits a solution $\mathbf{w}\in X^\sharp_{2+\alpha,\partial y}(\Omega_l)$.

Observe that $\mathbf{w}\in X^\sharp_{2+\alpha,\partial y}(\Omega_l)$ is a solution to the previous equation if and only if $\mathbf{w}\in D(L)$ and $L\mathbf{w} = \mathbf{f}-\mathcal{L}\mathbf{z}$, due to the Schauder type estimate in [8].

Since 0 is a semisimple eigenvalue of L, $\operatorname{Range} L = (I-P_0)(X^\sharp_0(\Omega_l))$. Consequently, the equation $(\mathcal{L}\mathbf{w},\mathcal{B}\mathbf{w}) = (\mathbf{f}-\mathcal{L}\mathbf{z},\mathbf{0})$ is solvable if and only if $P_0(\mathbf{f}-\mathcal{L}\mathbf{z}) = 0$, that is, if and only if (4.1) and (4.2) hold.

To conclude the proof let us check that the mapping $\widetilde{\mathcal{K}}$ is an isomorphism. $\widetilde{\mathcal{K}}$ is obviously bounded from $(I-P_0)(X^\sharp_{2+\alpha,\partial y}(\Omega_l))$ with values in $(I-\mathcal{R}_1-\mathcal{R}_2)(X^\sharp_\alpha(\Omega_l)\times C^{2+\alpha}_{\partial y}([-l,l])\times C^{1+\alpha}_{\partial y}([-l,l])\times C^{1+\alpha}_{\partial y}([-l,l]))$.

Let us prove that $\widetilde{\mathcal{K}}$ is onto. For this purpose we fix $(\mathbf{f},k_0,k_1,k_2)\in(I-\mathcal{R}_1-\mathcal{R}_2)(X^\sharp_\alpha(\Omega_l)\times C^{2+\alpha}_{\partial y}([-l,l])\times C^{1+\alpha}_{\partial y}([-l,l])\times C^{1+\alpha}_{\partial y}([-l,l]))$ and we observe that there exists $\mathbf{f}_0\in X^\sharp_\alpha(\Omega_l)$ such that $\mathbf{f} = \mathbf{f}_0-\mathcal{R}_1(\mathbf{f}_0,k_0,k_1,k_2)\mathbf{U}^0_x - \mathcal{R}_2(\mathbf{f}_0,k_0,k_1,k_2)\mathbf{U}^\lambda$. Hence $(\mathbf{f},k_0,k_1,k_2)$ belongs to the range of $\widetilde{\mathcal{K}}$ if and only if $P_0(\mathbf{f}-\mathcal{L}(\mathcal{M}(k_0,k_1)+\mathcal{N}k_2)) = 0$, that immediately follows from

$$\begin{aligned}
&P_0\big(\mathbf{f} - \mathcal{L}\big(\mathcal{M}(k_0, k_1) + \mathcal{N}k_2\big)\big) \\
&\quad = P_0\big(\mathbf{f}_0 - \mathcal{R}_1(\mathbf{f}_0, k_0, k_1, k_2)\mathbf{U}_x^0 - \mathcal{R}_2(\mathbf{f}_0, k_0, k_1, k_2)\mathbf{U}^\lambda \\
&\qquad - \mathcal{L}\big(\mathcal{M}(k_0, k_1) + \mathcal{N}k_2\big)\big) \\
&\quad = P_0\big(\mathbf{f}_0 - \mathcal{L}\big(\mathcal{M}(k_0, k_1) + \mathcal{N}k_2\big)\big) - P_0\big(\mathbf{f}_0 - \mathcal{L}\big(\mathcal{M}(k_0, k_1) + \mathcal{N}k_2\big)\big) \\
&\quad = 0.
\end{aligned}$$

Observe that $\widetilde{\mathcal{K}}$ is one to one. Indeed, $\widetilde{\mathcal{K}}(\mathbf{u}) = 0$ implies that $\mathbf{u} \in D(L)$ and $L\mathbf{u} = 0$. Hence $\mathbf{u}$ belongs to the kernel of L but, since $\mathbf{u} \in (I - P_0)(X^\sharp_{2+\alpha,\partial y}(\Omega_l))$, then $\mathbf{u} = \mathbf{0}$. $\square$

4.2. *Study of the nonlinear problem (1.8)*

This section is devoted to show the existence of nontrivial nonplanar TW solutions to (1.1) bifurcating from the planar TW solution (1.2) and to study the concavity of the bifurcating branches.

Proof of Theorem B (first part). The proof will be achieved adapting the technique of [4] to our situation.

Let us write problem (1.8) in the form

$$\mathcal{H}(\mathbf{u}, \mu, c) = (0, 0, 0), \tag{4.3}$$

where

$$\mathcal{H}: X^\sharp_{2+\alpha,\partial y}(\Omega_l) \times \mathbb{R}^2 \to X^\sharp_\alpha(\Omega_l) \times C^{2+\alpha}_{\partial y}\big([-l, l]\big) \times \big(C^{1+\alpha}_{\partial y}\big([-l, l]\big)\big)^2$$

is defined by

$$\begin{aligned}
\mathcal{H}(\mathbf{u}, \mu, c) = \big(&\mathcal{L}\mathbf{u} + \mathcal{F}_0\big(\mathbf{u}, \mu + \lambda(n)\big) - (c - 1)\big(\mathbf{U}_x^0 - v(0, \cdot)\mathbf{U}^0_{xx} + S_3\mathbf{u}_x\big), \\
&B_0\mathbf{u}, B_1\mathbf{u}, B_2\mathbf{u} - g(\mathbf{u})\big),
\end{aligned}$$

and $S_3\mathbf{u}$ is given in (1.8).

Taking the characterization of the weighted spaces $X^\sharp_k(\Omega_l)$ in Remark 2.3 into account, it is not hard to check that $\mathcal{F}_0(\mathbf{u}, \mu)$ belongs to $X^\sharp_\alpha(\Omega_l)$ so that $\mathcal{H}$ is well defined.

Let us observe that, if problem (4.3) admits a solution $(\mathbf{w}, \mu, c)$ belonging to $X^\sharp_{2+\alpha,\partial y}(\Omega_l) \times \mathbb{R}^2$, then it admits the infinitely many solutions $(\beta\mathbf{U}^0_x + \mathbf{w}, \mu, c)$, $\beta \in \mathbb{R}$. Indeed, for any $\mathbf{w} \in X^\sharp_{2+\alpha,\partial y}(\Omega_l)$ and any $\beta \in \mathbb{R}$ it holds $\mathcal{F}_0(\beta\mathbf{U}^0_x + \mathbf{w}) \equiv \mathcal{F}_0(\mathbf{w})$, $\mathcal{G}(\beta\mathbf{U}^0_x + \mathbf{w}) \equiv \mathcal{G}(\mathbf{w})$. Consequently,

$$\begin{aligned}
\mathcal{H}\big(\mathbf{w} + \beta\mathbf{U}^0_x, \mu, c\big) &= \mathcal{H}(\mathbf{w}, \mu, c) + (c - 1)\beta\big(\Theta^0_x(0)\mathbf{U}^0_{xx} - S_3\mathbf{U}^0_{xx}\big) \\
&= \mathcal{H}(\mathbf{w}, \mu, c).
\end{aligned}$$

Hence we just seek for a solution $(\mathbf{w},\mu,c)$ of (4.3) with $\mathbf{w}\in(I-P_1)(X^{\sharp}_{2+\alpha,\partial y}(\Omega_l))$, where $P_1\mathbf{u}=T_0(\mathbf{u})\mathbf{U}^0_x$ and T_0 is given by (3.12). For this purpose we are going to apply the Lyapunov–Schmidt method. Note that at $\lambda=\lambda(n)$ the kernel of $(\mathcal{L},B_0,B_1,B_2)=D_y\mathcal{H}(\mathbf{0},0,1)$ is two-dimensional and the codimension of its range is two, due to Theorem 4.1. But looking for a solution with $\mathbf{w}\in(I-P_1)(X^{\sharp}_{2+\alpha,\partial y}(\Omega_l))$ we reduce the dimension of the kernel to 1. Moreover, we shall reduce the number of unknowns and the codimension of the range by expressing c in terms of $\mathbf{w}$.

We begin by observing that the triplet $(\mathbf{w},\mu,c)$ is a solution of (4.3) if and only if

$$
\begin{aligned}
&\text{(a)}\quad \mathcal{R}_1\mathcal{H}(\mathbf{w},\mu,c)=0,\\
&\text{(b)}\quad (I-\mathcal{R}_1)\mathcal{H}(\mathbf{w},\mu,c)=0.
\end{aligned}
\tag{4.4}
$$

Let us write down explicitly (4.4a). For this purpose we observe that

$$
T_0(\mathcal{L}\mathbf{w})=\int_{-l}^{l}B_0\mathbf{w}(y)\,dy-\int_{-l}^{l}B_1\mathbf{w}(y)\,dy-\frac{1}{2l}\int_{-l}^{l}B_2\mathbf{w}(y)\,dy. \tag{4.5}
$$

Taking (4.5) into account, it can be checked that

$$
\begin{aligned}
\mathcal{R}_1\mathcal{H}(\mathbf{w},\mu,c)=\Bigg[&T_0\big(\mathcal{F}_0(\mathbf{w},\mu)\big)\\
&-(c-1)\left(1+\frac{1}{2l}\int_{-\infty}^{0}e^x\,dx\int_{-l}^{l}h(x,y)\,dy-\int_{-l}^{l}B_0\mathbf{w}(y)\,dy\right)\\
&-\frac{1}{2l}\int_{-l}^{l}g(\mathbf{w})(y)\,dy\Bigg]\big(\mathbf{U}^0_x,0,0,0\big).
\end{aligned}
\tag{4.6}
$$

Recalling that any solution $(\mathbf{w},\mu,c)$ to (4.3) is such that $B_2\mathbf{w}=0$, from (4.4a), taking (4.6) into account, we easily get

$$
\begin{aligned}
c-1=\mathcal{I}(\mathbf{w},\mu):=&\left(1+\frac{1}{2l}\int_{-\infty}^{0}e^x\,dx\int_{-l}^{l}h(x,y)\,dy\right)^{-1}\\
&\times\left(T_0\big(\mathcal{F}_0(\mathbf{w},\mu)\big)-\frac{1}{2l}\int_{-l}^{l}g(\mathbf{w})(y)\,dy\right),
\end{aligned}
\tag{4.7}
$$

provided the term in the first brackets does not vanish. This is certainly true if we take $\mathbf{w}$ in a sufficiently small neighborhood of 0.

Replacing $c-1$ into (4.4b) we immediately deduce that the triplet $(\mathbf{w}, \mu, c)$ is a solution to (4.3) if and only if $(\mathbf{w}, \mu)$ solves the equation $\widetilde{\mathcal{H}}(\mathbf{w}, \mu) = 0$, where

$$\widetilde{\mathcal{H}}(\mathbf{w}, \mu) = (I - \mathcal{R}_1)\big\{\mathcal{L}\mathbf{w} + \mathcal{F}_0(\mathbf{w}, \mu) - \mathcal{I}(\mathbf{w}, \mu)\big[\mathbf{U}^0_x - v(0, \cdot)\mathbf{U}^0_{xx} + S_3\mathbf{u}_x\big], \\ \mathcal{B}\mathbf{w} - \mathcal{G}(\mathbf{w})\big\}. \tag{4.8}$$

Obviously $\widetilde{\mathcal{H}}(0,0) = 0$. Moreover, $\widetilde{\mathcal{H}}$ is smooth and its Fréchet derivative with respect to $\mathbf{w}$ at $(0,0)$ is the operator (cf. (4.5))

$$\widetilde{\mathcal{H}}_{\mathbf{w}}(0,0) : (I - P_1)\big(X^{\sharp}_{2+\alpha,\partial y}(\Omega_l)\big) \to \\ (I - \mathcal{R}_1)\big(X^{\sharp}_{\alpha}(\Omega_l) \times C^{2+\alpha}_{\partial y}\big([-l,l]\big) \times C^{1+\alpha}_{\partial y}\big([-l,l]\big)\big)^2, \\ \widetilde{\mathcal{H}}_{\mathbf{w}}(0,0) = (I - \mathcal{R}_1)(\mathcal{L}, \mathcal{B}) = (\mathcal{L}, \mathcal{B}).$$

By Theorem 3.5, we know that the kernel of $\widetilde{\mathcal{H}}_{\mathbf{w}}(0,0)$ is one-dimensional and it is spanned by the function $\mathbf{U}^{\lambda}$ in (3.13). Moreover, Theorem 4.1 implies that the range of $\widetilde{\mathcal{H}}_{\mathbf{w}}(0,0)$ consists of all the quadruplets $(\mathbf{f}, k_0, k_1, k_2) \in (I - \mathcal{R}_1)(X^{\sharp}_{2+\alpha,\partial y}(\Omega_l) \times C^{2+\alpha}_{\partial y}([-l,l]) \times (C^{1+\alpha}_{\partial y}([-l,l]))^2)$ such that

$$\begin{aligned}\widetilde{\mathcal{R}}_2(\mathbf{f}, k_0, k_1, k_2) = T_{\lambda}\mathbf{f} &+ \rho \int_{-l}^{l} k_1(y) \cos\left(\frac{\pi n}{2l} y - \frac{\pi n}{2}\right) dy \\ &+ \sqrt{\lambda}\rho \int_{-l}^{l} k_2(y) \cos\left(\frac{\pi n}{2l} y - \frac{\pi n}{2}\right) dy \\ &- \left(\frac{\sqrt{\lambda}}{2} + \frac{1}{2}\right)\rho \int_{-l}^{l} k_0(y) \cos\left(\frac{\pi n}{2l} y - \frac{\pi n}{2}\right) dy = 0.\end{aligned} \tag{4.9}$$

In particular, the range of $\widetilde{\mathcal{H}}$ is one-dimensional.

We are going to apply the Crandall–Rabinowitz theorem about bifurcation for a simple eigenvalue (see [7]). For this purpose we have to check that the transversality condition

$$\widetilde{\mathcal{R}}_2\big(\widetilde{\mathcal{H}}_{\mu\mathbf{w}}(0,0)(\mathbf{U}^{\lambda})\big) \neq 0 \tag{4.10}$$

is fulfilled. As is easily seen, $\widetilde{\mathcal{H}}_{\mu\mathbf{w}}(0,0)\mathbf{v} = (0, -\Delta v, 0, v(0,\cdot), v(0,\cdot) + v_x(0,\cdot), 0)$, for any $\mathbf{v} \in X^{\sharp}_{2+\alpha}(\Omega_l)$ with sufficiently small norm. Thanks to (4.9), we get

$$\widetilde{\mathcal{R}}_2\big(\widetilde{\mathcal{H}}_{\mu\mathbf{w}}(0,0)(\mathbf{U}^{\lambda})\big) = \frac{1}{2} \frac{\sqrt{\lambda} - 1}{\lambda + 2\sqrt{\lambda} - 2}, \tag{4.11}$$

and (4.10) is trivially fulfilled since $\lambda = \lambda(n) = 1 + \pi^2 n^2 / l^2 > 1$. Hence the Crandall–Rabinowitz theorem implies that there exist $\delta_n > 0$, two smooth functions $\mu_n:(-\delta_n,\delta_n)\to\mathbb{R}$ and $\mathbf{w}_n:(-\delta_n,\delta_n)\to(I-P_1)(X^\sharp_{2+\alpha,\partial y}(\Omega_l))$ such that

$$\mu_n(0)=0,\qquad \mathbf{w}_n(0)=0,\qquad \mathbf{w}_n(\sigma)\neq 0\quad \text{for } \sigma\neq 0, \tag{4.12}$$

and

$$\widetilde{\mathcal{H}}\big(\mathbf{w}_n(\sigma),\mu_n(\sigma)\big)=0,\quad \forall\sigma\in(-\delta_n,\delta_n).$$

Moreover, there exists $\varepsilon_n>0$ such that if $|\mu|\leqslant\varepsilon_n$ and $\mathbf{w}\in B(0,\varepsilon)\subset(I-P_1)(X^\sharp_{2+\alpha,\partial y}(\Omega_l))$ are such that $\widetilde{\mathcal{H}}(\mathbf{w},\mu)=0$, then $\mu=\mu_n(\sigma)$ and $\mathbf{w}=\mathbf{w}_n(\sigma)$ for some $\sigma\in(-\delta_n,\delta_n)$.

Coming back to problem (4.3) it is now easy to check that there exist $(\mathbf{w}_n,\mu_n)$ as in (4.12) and a function $c_n:(-\delta_n,\delta_n)\to\mathbb{R}$ given by $c_n=1+\mathcal{I}(\mathbf{w}_n(\sigma),\mu_n(\sigma))$ such that $c_n(0)=1$ and

$$\mathcal{H}\big(\mathbf{w}_n(\sigma),\mu_n(\sigma),c_n(\sigma)\big)=0,\quad \forall\sigma\in(-\delta_n,\delta_n).$$

Moreover, there exists $\varepsilon_n>0$ such that if $|\mu|\leqslant\varepsilon_n$ and $\mathbf{w}\in B(0,\varepsilon)\subset(I-P_1)(X^\sharp_{2+\alpha,\partial y}(\Omega_l))$ are such that $\mathcal{H}(\mathbf{w},\mu,c)=0$, $c=c_n(\sigma)$, then $\mu=\mu_n(\sigma)$ and $\mathbf{w}=\mathbf{w}_n(\sigma)$ for some $\sigma\in(-\delta_n,\delta_n)$.

Let us prove that $\mathbf{w}_n(\sigma)$ depends explicitly on y, if $\sigma\neq 0$. Without losing in generality we can suppose that $c_n(\sigma)>0$ for any $\sigma\in(-\delta_n,\delta_n)$. A straightforward computation shows that if $\mathcal{H}(\mathbf{w},\mu,c)=0$ with $\mathbf{w}$ independent of y, and $c>0$, then $\mathbf{w}=k\mathbf{U}^0_x$ for some constant $k\in\mathbb{R}$, and $c=1$. Since $\mathbf{w}_n(\sigma)\in(I-P_1)(X^\sharp_{2+\alpha}(\Omega_l))$ it follows that $\mathbf{w}_n(\sigma)=(I-P_1)(k\mathbf{U}^0_x)=0$, that leads us to a contradiction, since $\mathbf{w}_n(\sigma)\neq 0$ if $\sigma\neq 0$.

Coming back to problem (1.1), from the previous results and performing the inverse transformations that led us to problem (1.8) (see also [8, Section 2]), we deduce that for any $\sigma\in(-\delta_n,\delta_n)$ there exists a quadruplet $(\mathbf{U}_n(\sigma),\xi^1_n(\sigma),c_n(\sigma),\lambda_n(\sigma))$ satisfying the assertion of Theorem B.

The proof of the first part of Theorem B is now complete. □

To conclude this section we study the concavity of the bifurcation branches and the behaviour of c_n in a neighborhood of $\sigma=0$ and we prove the second part of Theorem B. As far as c_n is concerned, it is quite easy to show that it is locally greater than 1 in a neighborhood of $\sigma=0$. Things are much more difficult when dealing with the concavity of λ_n. In fact, for any $n\in\mathbb{N}$ the bifurcated branches are parabolic near $\lambda(n)$. We will show that for any $l\in\mathbb{R}_+$ there exists a critical value $n_0=n_0(l)\in\mathbb{N}$ such that if $n\geqslant n_0$, then $\lambda_n''(0)>0$. Throughout the rest of this section the symbol $'$ will denote differentiation with respect to σ.

Theorem 4.2. *For any $l\in\mathbb{R}_+$ and any $n\in\mathbb{N}$, $\mu_n'(0)=0$ and there exists $n_0=n_0(l)\in\mathbb{N}$ such that $\mu_n''(0)>0$ for any $n\geqslant n_0$. Moreover, the functions c_n are locally greater than* 1 *in a neighborhood of* 0 *for any $n\in\mathbb{N}$.*

Proof. Here we prove only the last part of the theorem, postponing the proof of the first part until Appendix B, since it relies essentially on heavy computations.

We recall that for any $\sigma \in (-\delta_n, \delta_n)$

$$c_n(\sigma) - 1 = \mathcal{I}\big(\mathbf{w}_n(\sigma), \mu_n(\sigma)\big), \tag{4.13}$$

where $\mathcal{I}$ is given by (4.7). Differentiating (4.13) with respect to σ, evaluating the so obtained equality at $\sigma = 0$, and recalling that from the proof of [7, Theorem 1.7] it follows that $\mathbf{w}_n'(0) = \mathbf{U}^\lambda$, we get $c_n'(0) = \mu_n'(0)\mathcal{I}_\mu(0,0) + \mathcal{I}_{\mathbf{w}}(0,0)\mathbf{U}^\lambda$. Since $\mathcal{I}_{\mathbf{w}}(0,0) = 0$, $\mathcal{I}_\mu(0,0) = 0$, we obtain $c_n'(0) = 0$.

Then we twice differentiate (4.13) with respect to σ and evaluate both sides at $\sigma = 0$. Observing that $\mathcal{I}_{\mathbf{w}\mu}(0,0) = 0$, $\mathcal{I}_{\mu\mu}(0,0) = 0$, we deduce that $c_n''(0) = \mathcal{I}_{\mathbf{ww}}(0,0)(\mathbf{U}^\lambda, \mathbf{U}^\lambda)$. Straightforward computations show that

$$\begin{aligned}\mathcal{I}_{\mathbf{ww}}(0,0)(\mathbf{U}^\lambda, \mathbf{U}^\lambda) &= T_0\big(D_{\mathbf{ww}}\mathcal{F}_0(0,0)(\mathbf{U}^\lambda, \mathbf{U}^\lambda)\big) \\ &\quad - \frac{1}{2l}\int_{-l}^{l} g_{\mathbf{ww}}(0)(\mathbf{U}^\lambda, \mathbf{U}^\lambda)(y)\,dy \\ &= \frac{1}{4}\big(\sqrt{\lambda} - 1\big)\big(2\sqrt{\lambda} - 1\big).\end{aligned}$$

Consequently, $c_n''(0) > 0$ for any $\lambda > 1$. Since $\lambda = \lambda(n) = 1 + \pi^2 n^2/l^2$, $n \in \mathbb{N} \setminus \{0\}$, we immediately deduce the assertion. $\square$

Appendix A. Asymptotic behaviour in the linear problem

Throughout this section $\sigma_+(L)$ denotes the set of all $\lambda \in \sigma(L)$ with positive real parts, P^+ the projection defined by (3.9).

Let us consider the backward problem

$$\begin{cases} D_t\mathbf{u}(t,\cdot) = \mathcal{L}\mathbf{u}(t,\cdot) + \mathbf{f}(t,\cdot), & t \leqslant 0, \\ \mathcal{B}\mathbf{u}(t,\cdot) = \psi(t,\cdot), & t \leqslant 0, \\ D_y\mathbf{u}(t,\cdot) = \mathbf{0}, & t \leqslant 0, \\ \mathbf{u}(0,\cdot) = \mathbf{u}_0. \end{cases} \tag{A.1}$$

We introduce the lifting operator $\mathcal{N}$ defined by $\mathcal{N}\psi = (-N\psi, -\lambda N\psi, 0)$ for any $\psi \in C([-l,l])$, where

$$N\psi(x,y) = \eta(x)x\int_{\mathbb{R}} \varphi(\xi)E\psi(y + \xi x)\,d\xi, \quad (x,y) \in \overline{\Omega}_l. \tag{A.2}$$

$\varphi \in C_0^\infty(\mathbb{R})$ is any nonnegative even function compactly supported in $B(0,1)$ and such that $\|\varphi\|_{L^1(\mathbb{R})} = 1$, while η is any infinitely many differentiable function such that $\eta \equiv 1$ in $[-1,0]$ and $\eta \equiv 0$ in $(-\infty,-2]$. $E \in L(C(\overline{\Omega}_l), C(\mathbb{R}^2_-))$ is the extension operator defined as follows: $E\psi(x,y) = E\psi(x,(-1)^n(y - 2ln))$ for any $x \leqslant 0$, any $y \in [(2n-1)l, (2n+1)l)$ and any $n \in \mathbb{Z}$ (see also [8, Theorem 4.1, Lemma A.6]).

Lemma A.1. *Suppose that* $\lambda \in A_l$. *Fix* $0 < \omega_0 < m_+ := \min\{\mathrm{Re}\,\omega\colon\ \omega \in \sigma_+(L)\}$. *Suppose that* $e^{-\omega_0 t}\mathbf{f} \in \mathcal{X}_{\alpha/2,\alpha}(-\infty, 0, \Omega_l)$, $e^{-\omega_0 t}\psi \in C^{(1+\alpha)/2,1+\alpha}((-\infty, 0] \times [-l, l])$, *with* $D_y\psi(\cdot, \pm l) = 0$, *and* $\mathbf{u}_0 \in X_{2+\alpha}(\Omega_l)$. *Then problem* (A.1) *admits a solution* $\mathbf{u} \in \mathcal{X}_{1+\alpha/2,2+\alpha}(-T, 0, \Omega_l)$ *for any* $T \in \mathbb{R}_-$, *with* $\mathbf{v} := e^{-\omega_0 t}\mathbf{u}$ *belonging to* $B((-\infty, 0]; X_0(\Omega_l))$, *if and only if*

$$(I - P^+)\mathbf{u}_0 = \int_{-\infty}^{0} e^{-sL}(I - P^+)\big[\mathbf{f}(s, \cdot) + \mathcal{L}\mathcal{N}\psi(s, \cdot)\big]\,ds - L\int_{-\infty}^{0} e^{-sL}(I - P^+)\mathcal{N}\psi(s, \cdot)\,ds. \tag{A.3}$$

In such a case, $\mathbf{u}$ *is given by*

$$\begin{aligned}\mathbf{u}(t, \cdot) = {} & e^{tL}P^+\mathbf{u}_0 + \int_0^t e^{(t-s)L}P^+\big[\mathbf{f}(s, \cdot) + \mathcal{L}\mathcal{N}\psi(s, \cdot)\big]\,ds \\ & - L\int_0^t e^{(t-s)L}P^+\mathcal{N}\psi(s, \cdot)\,ds \\ & + \int_{-\infty}^{t} e^{(t-s)L}(I - P^+)\big[\mathbf{f}(s, \cdot) + \mathcal{L}\mathcal{N}\psi(s, \cdot)\big]\,ds \\ & - L\int_{-\infty}^{t} e^{(t-s)L}(I - P^+)\mathcal{N}\psi(s, \cdot)\,ds, \\ & t \in (-\infty, 0]. \end{aligned}\tag{A.4}$$

Moreover, $e^{-\omega_0 t}\mathbf{u} \in \mathcal{X}_{1+\alpha/2,2+\alpha}(-\infty, 0, \Omega_l)$ *and there exists a positive constant* C *such that*

$$\begin{aligned}&\|e^{-\omega_0 t}\mathbf{u}\|_{\mathcal{X}_{1+\alpha/2,2+\alpha}(-\infty,0,\Omega_l)} \\ &\quad \leqslant C\big(\|\mathbf{u}_0\|_{X_0(\Omega_l)} + \|e^{-\omega_0 t}\mathbf{f}\|_{\mathcal{X}_{\alpha/2,\alpha}(-\infty,0,\Omega_l)} \\ &\qquad + \|e^{-\omega_0 t}\psi\|_{C^{(1+\alpha)/2,1+\alpha}((-\infty,0]\times[-l,l])}\big).\end{aligned}\tag{A.5}$$

Proof. The proof can be obtained arguing as in the proof of [2, Theorem 0.2] taking the Schauder estimates in [8] into account. Hence it is omitted. □

Remark A.2. The previous lemma holds also in the weighted case. The lifting operator $\mathcal{N}$ can be used also in the weighted case. Observe in particular that $\sigma_+(L) \subset \sigma_{\mathrm{point}}(L)$. Indeed, $\sigma^\sharp(L) = \sigma^\sharp_{\mathrm{cont}}(L) \cup \sigma_{\mathrm{point}}(L)$ (see (3.1)), and $\sigma^\sharp_{\mathrm{cont}}(L)$ is contained in the half space $\mathrm{Re}\,\omega < 0$.

We now deal with the problem

$$\begin{cases} D_t\mathbf{u}(t,\cdot) = (I - P_0)\mathcal{L}\mathbf{u}(t,\cdot) + \mathbf{f}(t,\cdot), & t \geqslant 0, \\ \mathcal{B}\mathbf{u}(t,\cdot) = \psi(t,\cdot), & t \geqslant 0, \\ D_y\mathbf{u}(t,\cdot) = \mathbf{0}, & t \geqslant 0, \\ \mathbf{u}(0,\cdot) = \mathbf{u}_0, \end{cases} \tag{A.6}$$

where P_0 is the spectral projection associated with the eigenvalue $\omega = 0$ and $X = (I - P_0)(X^\sharp_0(\Omega_l))$.

Taking the Schauder estimates in [8] into account and arguing as in the proof of [2, Theorem 0.1] we can show the following result.

Lemma A.3. *Suppose that* $\lambda \notin \overline{A}_l$ *and fix* $\omega_0 \in (0, -\max\{\mathrm{Re}\,\omega\colon \omega \in \sigma(L)\})$. *Then for any triplet of functions* $\mathbf{u}_0 \in X^\sharp_{2+\alpha}(\Omega_l) \cap X$, $e^{\omega_0 t}\mathbf{f} \in \mathcal{X}^\sharp_{\alpha/2,\alpha}(0,+\infty,\Omega_l) \cap B([0,+\infty); X)$ *and* $e^{\omega_0 t}\psi \in C^{(1+\alpha)/2,1+\alpha}([0,+\infty) \times [-l,l]) \cap B([0,+\infty); X)$, *satisfying the compatibility conditions*

$$\mathcal{B}(\mathbf{u}_0) = \big(0, 0, \psi(0,\cdot)\big), \qquad B_0\big[\mathcal{L}\mathbf{u}_0 + \mathbf{f}(0,\cdot)\big] = 0,$$
$$D_y\mathbf{u}_0(\cdot, \pm l) = \mathbf{0}, \qquad D_y\psi(\cdot, \pm l) = 0,$$

problem (A.6) *admits a unique solution* $\mathbf{u}$ *belonging to* $\mathcal{X}^\sharp_{1+\alpha/2,2+\alpha}(0,T,\Omega_l)$ *for any* $T > 0$ *and such that* $e^{\omega_0 t}\mathbf{u}$ *belongs to* $\mathcal{X}^\sharp_{1+\alpha/2,2+\alpha}(0,+\infty,\Omega_l) \cap B([0,+\infty); X)$, *given by*

$$\begin{aligned} \mathbf{u}(t,\cdot) = e^{tL}\mathbf{u}_0 &+ \int_0^t e^{(t-s)L}\big(\mathbf{f}(s,\cdot) + \widetilde{\mathcal{L}}\mathcal{N}\psi(s,\cdot)\big)\,ds \\ &- L\int_0^t e^{(t-s)L}(I - P_0)\mathcal{N}\psi(s,\cdot)ds, \quad t \geqslant 0, \end{aligned} \tag{A.7}$$

where $\widetilde{\mathcal{L}} = (I - P_0)\mathcal{L}$ *and* $\mathcal{N}$ *is the lifting operator defined at the beginning of Appendix A. Moreover, there exists a positive constant* C, *independent of data, such that*

$$\begin{aligned} &\|e^{\omega_0 t}\mathbf{u}\|_{\mathcal{X}^\sharp_{1+\alpha/2,2+\alpha}(0,+\infty,\Omega_l)} \\ &\quad\leqslant C\big(\|\mathbf{u}_0\|_{X^\sharp_{2+\alpha}(\Omega_l)} + \|e^{\omega_0 t}\mathbf{f}\|_{\mathcal{X}^\sharp_{\alpha/2,\alpha}(0,+\infty,\Omega_l)} \\ &\qquad + \|e^{\omega_0 t}\psi\|_{C^{(1+\alpha)/2,1+\alpha}([0,+\infty)\times[-l,l])}\big). \end{aligned} \tag{A.8}$$

Appendix B. Determining the concavity of the branches

This section is devoted to determine the concavity of the function λ_n ($n \in \mathbb{N}$) in a neighborhood of $\sigma = 0$.

Proof of the first part of Theorem 4.2. As is easily seen we can merely deal with the function μ_n since λ_n and μ_n differ by a constant.

Observe that the couple $(\mathbf{w}_n(\sigma), \mu_n(\sigma))$ defined in the proof of Theorem B solves the equation

$$\overline{\mathcal{H}}\big(\mathbf{w}_n(\sigma), \mu_n(\sigma)\big) = \mathbf{0}, \quad \sigma \in (-\delta_n, \delta_n), \tag{B.1}$$

where

$$\begin{aligned}\overline{\mathcal{H}}(\mathbf{w}, \mu) = \big(&\mathcal{L}\mathbf{w} + \mathcal{F}_0(\mathbf{w}, \mu) - \mathcal{I}(\mathbf{w}, \mu)\big(\mathbf{U}_x^0 - v(0, \cdot)\mathbf{U}_{xx}^0 + S_3\mathbf{u}_x\big),\\ &\mathcal{B}\mathbf{w} - \mathcal{G}(\mathbf{w})\big).\end{aligned}$$

Differentiating both sides of (B.1) with respect to σ and evaluating them at $\sigma = 0$ gives $\overline{\mathcal{H}}_\mu(\mathbf{0}, 0)\mu_n'(0) + \overline{\mathcal{H}}_{\mathbf{w}}(\mathbf{0}, 0)(\mathbf{U}^\lambda) = \mathbf{0}$. Since $D_\mu^k \overline{\mathcal{H}}(\mathbf{0}, 0) = \mathbf{0}$, for any $k \in \mathbb{N}$, we cannot make $\mu'(0)$ explicit in terms of the other quantities. Hence we need to twice differentiate (B.1) with respect to σ and evaluate the so obtained function at $\sigma = 0$. We get

$$2\mu_n'(0)\overline{\mathcal{H}}_{\mu\mathbf{w}}(0, 0)(\mathbf{U}^\lambda) + \overline{\mathcal{H}}_{\mathbf{ww}}(0, 0)(\mathbf{U}^\lambda, \mathbf{U}^\lambda) = -\overline{\mathcal{H}}_{\mathbf{w}}(0, 0)\mathbf{w}''(0). \tag{B.2}$$

Observe that the right-hand side in (B.2) belongs to the range of $(\mathcal{L}, \mathcal{B})$. Hence, applying $\widetilde{\mathcal{R}}_2$ (cf. (4.9)) to both the sides of (B.2), and observing that $\widetilde{\mathcal{R}}_2(\overline{\mathcal{H}}_{\mu\mathbf{w}}(\mathbf{0}, 0)(\mathbf{U}^\lambda)) = \widetilde{\mathcal{R}}_2(\widetilde{\mathcal{H}}_{\mu\mathbf{w}}(\mathbf{0}, 0)(\mathbf{U}^\lambda)) > 0$ (cf. (4.7)), thanks to (4.11), we deduce that

$$\mu_n'(0) = -\frac{1}{2}\frac{\widetilde{\mathcal{R}}_2(\overline{\mathcal{H}}_{\mathbf{ww}}(\mathbf{0}, 0)(\mathbf{U}^\lambda, \mathbf{U}^\lambda))}{\widetilde{\mathcal{R}}_2(\overline{\mathcal{H}}_{\mu\mathbf{w}}(\mathbf{0}, 0)(\mathbf{U}^\lambda))}.$$

Long but straightforward computations show that $\widetilde{\mathcal{R}}_2(\overline{\mathcal{H}}_{\mathbf{ww}}(\mathbf{0}, 0)(\mathbf{U}^\lambda, \mathbf{U}^\lambda)) = 0$. Hence $\mu_n'(0) = 0$ (for a detailed proof see [9]).

Hence to determine the behaviour of μ_n in a neighborhood of $\sigma = 0$ we need to consider the second-order derivative of such a function. Differentiating thrice with respect to σ the equation $\overline{\mathcal{H}}(\mu_n(\sigma), \mathbf{w}_n(\sigma)) = 0$ and evaluating the so obtained function at $\sigma = 0$, we finally deduce that

$$\begin{aligned}&\overline{\mathcal{H}}_{\mathbf{www}}(\mathbf{0}, 0)(\mathbf{U}^\lambda, \mathbf{U}^\lambda, \mathbf{U}^\lambda) + 3\overline{\mathcal{H}}_{\mathbf{ww}}(\mathbf{0}, 0)\big(\mathbf{U}^\lambda, \mathbf{w}_n''(0)\big)\\ &\quad = -\overline{\mathcal{H}}_{\mathbf{w}}(\mathbf{0}, 0)\mathbf{w}_n''(0),\end{aligned} \tag{B.3}$$

so that the left-hand side of (B.3) belongs to the range of the operator $(\mathcal{L}, \mathcal{B})$. Consequently, applying the functional $\mathcal{R}_2$ to both the sides of (B.3) we obtain an explicit expression for $\mu_n''(0)$, that is,

$$\mu_n''(0) = -\frac{\widetilde{\mathcal{R}}_2(\overline{\mathcal{H}}_{\mathbf{ww}}(\mathbf{0},0)(\mathbf{U}^\lambda, \mathbf{w}_n''(0)))}{\widetilde{\mathcal{R}}_2(\overline{\mathcal{H}}_{\mu\mathbf{w}}(\mathbf{0},0)(\mathbf{U}^\lambda))} - \frac{1}{3}\frac{\widetilde{\mathcal{R}}_2(\overline{\mathcal{H}}_{\mathbf{www}}(\mathbf{0},0)(\mathbf{U}^\lambda, \mathbf{U}^\lambda, \mathbf{U}^\lambda))}{\widetilde{\mathcal{R}}_2(\overline{\mathcal{H}}_{\mu\mathbf{w}}(\mathbf{0},0)(\mathbf{U}^\lambda))}. \tag{B.4}$$

To compute $\mu_n''(0)$ we need to determine the function $\mathbf{w}_n''(0)$. From (B.2) we deduce that $\mathbf{w}_n''(0)$ is determined via the equation $\overline{\mathcal{H}}_{\mathbf{ww}}(\mathbf{0},0)(\mathbf{U}^\lambda, \mathbf{U}^\lambda) = -\overline{\mathcal{H}}_{\mathbf{w}}(\mathbf{0},0)\mathbf{w}_n''(0)$. Hence to determine $\mathbf{w}_n''(0)$ we have to solve the problem

$$\begin{cases} \mathcal{L}\mathbf{w} = \mathbf{f}, & \text{in } \overline{\Omega}_l, \\ \mathcal{B}\mathbf{w} = \mathbf{g}, & \text{in } [-l,l], \\ D_y\mathbf{w}(\cdot, \pm l) = \mathbf{0}, & \text{in } (-\infty, 0], \end{cases} \tag{B.5}$$

where $(\mathbf{f},\mathbf{g}) = \overline{\mathcal{H}}_{\mathbf{ww}}(\mathbf{0},0)(\mathbf{U}^\lambda, \mathbf{U}^\lambda)$, looking for a smooth solution $\mathbf{w} = (v,w,h)$.

Since $\overline{\mathcal{H}}_{\mathbf{ww}}(\mathbf{0},0)(\mathbf{U}^\lambda, \mathbf{U}^\lambda)$ is split into the sum of two functions, the first one independent of y and the latter one given by $\mathbf{h}(x)\cos(\pi ny/l - \pi n)$ for some smooth function $\mathbf{h}$, it is natural to look for a solution to (B.5) in the form $\mathbf{w} = \mathbf{w}^{(1)}(x) + \mathbf{w}^{(2)}(x)\cos(\pi ny/l - \pi n)$. Long but easy computations show that a suitable function $\mathbf{w}^{(1)} = (v_1, w_1, h_1)$ is given by

$$\begin{aligned} v_1(x) &= \frac{1}{4}(\lambda^{1/2}-1)(2\lambda^{1/2}-1)xe^x - \frac{1}{2}(\lambda^{1/2}+1)e^{(\sqrt{\lambda}+1)x/2}, \\ w_1(x) &= -\frac{1}{4}\big[\lambda^{1/2}(\lambda^{1/2}+1)^2x + (\lambda^{1/2}+1)(2\lambda+3\lambda^{1/2}-1)\big]e^{(\sqrt{\lambda}+1)x/2} \\ &\quad + \frac{1}{4}\lambda(\lambda^{1/2}-1)(2\lambda^{1/2}-1)(x^2+x)e^x + \lambda e^x, \\ h_1(x) &= \frac{1}{4}(\lambda^{1/2}-1)^2 e^{(\sqrt{\lambda}-1)x/2}, \end{aligned}$$

and a suitable function $\mathbf{w}_2 = (v_2, w_2, h_2)$ is given by

$$\begin{aligned} v_2(x) &= \frac{\lambda^{1/2}(4\lambda-3)^{1/2}[(2\lambda^{1/2}-3)(4\lambda-3)^{1/2}+\lambda^{1/2}]}{6(\lambda-1)[(4\lambda-3)^{1/2}-1]}e^{(\sqrt{4\lambda-3}+1)x/2} \\ &\quad + \frac{1}{2}e^x - \frac{1}{2}(\lambda^{1/2}+1)e^{(\sqrt{\lambda}+1)x/2}, \\ w_2(x) &= -\frac{1}{4}\big[\lambda^{1/2}(\lambda^{1/2}+1)^2x + (\lambda^{1/2}+1)(2\lambda+3\lambda^{1/2}-1)\big]e^{(\sqrt{\lambda}+1)x/2} \\ &\quad + \frac{1}{2}\frac{\lambda^{3/2}[(4\lambda-3)^{1/2}+1]^2[(2\lambda^{1/2}-3)(4\lambda-3)^{1/2}+\lambda^{1/2}]}{24(\lambda-1)^2} \\ &\quad \times xe^{(\sqrt{4\lambda-3}+1)x/2} + \frac{1}{2}\lambda(x+2)e^x \\ &\quad + \lambda\left[\frac{(7\lambda^{1/2}+4)(4\lambda-3)^{1/2}+16\lambda^{3/2}-8\lambda-21\lambda^{1/2}}{12(\lambda^{1/2}+1)((4\lambda-3)^{1/2}-1)}\right] \\ &\quad \times e^{(\sqrt{4\lambda-3}+1)x/2}, \end{aligned}$$

$$h_2(x) = \lambda\left[\frac{(4\lambda-3)^{1/2} - \lambda - 3\lambda^{1/2} + 3}{12(\lambda-1)}\right]e^{(\sqrt{4\lambda-3}-1)x/2} + \frac{1}{4}(\lambda^{1/2}-1)^2 e^{(\sqrt{\lambda}-1)x/2}.$$

So, we have determined explicitly a solution $\mathbf{w}_n^{(0)}$ to (B.5). This is enough for our aims. Indeed, if $\mathbf{v}$ is another solution, the difference $\mathbf{v} - \mathbf{w}_n^{(0)}$ belongs to $\operatorname{Ker} L$. Since $\mathbf{w}_n(\sigma)$ belongs to $(I - P_1)(X^\sharp_{2+\alpha}(\Omega_l))$ for every σ, then $\mathbf{w}_n''(0)$ does and, therefore, $\mathbf{w}_n''(0) = \mathbf{w}_n^{(0)} + c\mathbf{U}^\lambda$ for some real constant c. It follows that

$$\begin{aligned}
&\widetilde{\mathcal{R}}_2\big(\overline{\mathcal{H}}_{\mathbf{ww}}(\mathbf{0},0)\big(\mathbf{U}^\lambda, \mathbf{w}_n''(0)\big)\big) \\
&\quad = \widetilde{\mathcal{R}}_2\big(\overline{\mathcal{H}}_{\mathbf{ww}}(\mathbf{0},0)\big(\mathbf{U}^\lambda, \mathbf{w}_n^{(0)}\big)\big) + c\widetilde{\mathcal{R}}_2\big(\overline{\mathcal{H}}_{\mathbf{ww}}(\mathbf{0},0)(\mathbf{U}^\lambda, \mathbf{U}^\lambda)\big) \\
&\quad = \widetilde{\mathcal{R}}_2\big(\overline{\mathcal{H}}_{\mathbf{ww}}(\mathbf{0},0)\big(\mathbf{U}^\lambda, \mathbf{w}_n^{(0)}\big)\big),
\end{aligned}$$

which has to be replaced in (B.4).

We are now in a position to compute $\mu_n''(0)$ from (B.4). Long but straightforward computations show that

$$\begin{aligned}
\mu_n''(0) &= \mu_n''(0,\lambda) \\
&= \big[(4\lambda-3)^{1/2}\big(87\lambda^4 - 522\lambda^{7/2} + 696\lambda^3 + 808\lambda^{5/2} - 1296\lambda^2 \\
&\quad - 396\lambda^{3/2} + 564\lambda + 66\lambda^{1/2} - 63\big) \\
&\quad + 24\lambda^{9/2} - 999\lambda^4 + 1722\lambda^{7/2} + 1876\lambda^3 - 3132\lambda^{5/2} - 852\lambda^2 \\
&\quad + 1680\lambda^{3/2} - 60\lambda - 234\lambda^{1/2} + 63\big] \\
&\quad \times \big\{48(\lambda^{1/2}+1)\big[(4\lambda-3)^{1/2} + \lambda^{1/2}\big]^2\big[(4\lambda-3)^{1/2} - 1\big]\big\}^{-1}.
\end{aligned}$$

Observe that the sign of $\mu_n''(0)$ is determined by the sign of the term in the first square brackets (say g) since the other term is positive when $\lambda \in [1,+\infty)$. To determine the sign of g we begin by observing that the function g_1 defined by

$$g_1(\lambda) = 87\lambda^4 - 522\lambda^{7/2} + 696\lambda^3 + 808\lambda^{5/2} - 1296\lambda^2 - 396\lambda^{3/2} + 564\lambda + 66\lambda^{1/2} - 63$$

is nonnegative for $\lambda \geqslant 2$ and $(4\lambda-3)^{1/2} \geqslant (2\lambda^{1/2}-1)$ for any $\lambda \geqslant 1$. Hence

$$\begin{aligned}
g(\lambda^2) &\geqslant 198\lambda^9 - 2130\lambda^8 + 3636\lambda^7 + 2796\lambda^6 - 6532\lambda^5 - 348\lambda^4 \\
&\quad + 3204\lambda^3 - 492\lambda^2 - 426\lambda + 126,
\end{aligned} \tag{B.6}$$

and the function in the right-hand side of (B.6) is positive increasing in $[9,+\infty)$. Consequently, we can limit ourselves to studying $\mu_n''(0,\cdot)$ in $[1,81)$. A direct analysis of $\mu_n''(0,\cdot)$ shows that it is positive in $(1,\lambda_0)$ and in $(\lambda_1, 9)$, where $\lambda_0 \in (4.33440627, 4.33440628)$ and $\lambda_1 \in (67.53976333, 67.539763335)$ and

$$\mu_n''(0, 4.33440627) \sim 0.00109677292193,$$
$$\mu_n''(0, 4.33440628) \sim -0.00203195797718,$$
$$\mu_n''(0, 67.53976333) \sim -0.81658303246377,$$
$$\mu_n''(0, 67.539763335) \sim 0.09168112950397.$$

Coming back to our problem, we deduce that for any $l \in \mathbb{R}_+$ there exists $n_0 = n_0(l) \in \mathbb{N}$ such that for any $n \geqslant n_0$, $\mu_n''(0) > 0$, and the proof is now complete. $\square$

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